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CONTENTS

G. H. Hardy and **W. W. Rogosinski**: Notes on Fourier Series (III): Asymptotic Formulae for the Sums of Certain Trigonometrical Series . 49

A. P. Guinand: Gauss Sums and Primitive Characters 59

W. H. J. Fuchs: A Theorem on Hausdorff's Methods of Summation 64

S. D. Daymond: Some Solutions of the Hodograph Equation governing the Two-Dimensional Flow of a Compressible Fluid . . . 78

A. S. Besicovitch: On the Definition and Value of the Area of a Surface 86

E. C. Titchmarsh: On Expansions in Eigenfunctions
VII 103
VIII 115

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NOTES ON FOURIER SERIES (III): ASYMPTOTIC FORMULAE FOR THE SUMS OF CERTAIN TRIGONOMETRICAL SERIES

By G. H. HARDY (*Cambridge*) and
W. W. ROGOSINSKI (*Aberdeen*)

[Received 21 October 1943]

1. OUR object in this note is to find asymptotic formulae for the sums of the series

$$\mathbf{C} = \mathbf{C}(x) = \sum a_n \cos nx, \qquad \mathbf{S} = \mathbf{S}(x) = \sum b_n \sin nx, \tag{1.1}$$

where a_n and b_n behave approximately like a power $n^{-\alpha}$, when $x \to 0$. We shall suppose that a_n and b_n are of the form $n^{-\alpha}\phi_n$, where

$$\alpha = \beta + i\gamma, \tag{1.2}$$

and ϕ_n is a function, in one sense or another, of a 'slowly oscillating' type; and that $0 < \beta < 1$ or $0 < \beta \leqslant 1$ according as we are dealing with **C** or **S**. The case $\beta = 0$ is exceptional for **S**, and both $\beta = 0$ and $\beta = 1$ for **C**: we reserve these cases for later consideration. We state our proofs primarily for **S**, since in this case we can allow β to be 1. Our main results are

THEOREM 1. *Suppose that*

$$b_n = n^{-\alpha}\phi_n, \tag{1.3}$$

where $\alpha = \beta + i\gamma$, $0 < \beta \leqslant 1$; *and that* ϕ_n *satisfies*

(A) $$\phi_1 + \phi_2 + \ldots + \phi_n \sim n\phi_n,$$

and **one** *of the three conditions*

(B 1) ϕ_n *is real and* $|b_n|$ *decreases*;

(B 2) ϕ_n *is real and increasing*;

(B 3) $$\phi_{n+1}/\phi_n = 1 + O(n^{-1}).$$

Then $\mathbf{S}(x) = \sum b_n \sin nx$ *is convergent, and*

$$\mathbf{S}(x) \sim \xi b_\xi J = \xi^{1-\alpha}\phi_\xi J \tag{1.4}$$

when $x \to +0$, *where* $\xi = [x^{-1}]$,

$$J = \int_0^\infty t^{-\alpha} \sin t \, dt = \Gamma(1-\alpha)\cos \tfrac{1}{2}\alpha\pi \tag{1.5}$$

when $\alpha \neq 1$, *and* $J = \frac{1}{2}\pi$ *when* $\alpha = 1$. *In particular, either of the conditions*

(C 1) ϕ_n *is real and* $n^{\delta}\phi_n$ *increasing,* $n^{-\delta}\phi_n$ *decreasing, for every positive* δ,

(C 2) $$\phi_{n+1}/\phi_n = 1+o(n^{-1}),$$

is in itself sufficient for (1.4).

It should be observed that a real ϕ_n which satisfies (A) is ultimately of fixed sign.

THEOREM 2. *There are corresponding results for* $\mathbf{C}(x)$ *when* $0 < \beta < 1$, *with* J *replaced by the corresponding cosine integral.*

It is not obvious in either case, except when we adopt hypothesis (B 1), that the series are convergent.

2. It will be convenient to begin by proving the results in the special case $\phi_n = 1$. They are then familiar (at any rate for real α); but the usual proofs follow different lines, and it will elucidate our method and shorten our later analysis if we insert one here.

LEMMA 1. *If* γ *is real and fixed then*

$$\sum_{\mu}^{\nu} n^{-i\gamma}e^{nix} = O\left(\frac{1}{|x|}\right)$$

when $x \to 0$, *uniformly in* μ *and* ν.

We suppose that $x > 0$ and $\mu \leqslant \xi = [x^{-1}] < \nu$, and write

$$T = \sum_{\mu}^{\nu} n^{-i\gamma}e^{nix} = \sum_{\mu}^{\xi} + \sum_{\xi+1}^{\nu} = T_1 + T_2.$$

Plainly $T_1 = O(x^{-1})$. Also

$$(1-e^{ix})T_2 = \sum_{\xi+1}^{\nu} n^{-i\gamma}\Delta e^{nix} = O(1) - \sum_{\xi+1}^{\nu} e^{nix}\Delta(n-1)^{-i\gamma} = O(1) - T_3,$$

$$(1-e^{ix})T_3 = \sum_{\xi+1}^{\nu} \Delta(n-1)^{-i\gamma}\Delta e^{nix} = O\left(\frac{1}{\xi}\right) - \sum_{\xi+1}^{\nu} e^{nix}\Delta^2(n-2)^{-i\gamma}$$

$$= O\left(\frac{1}{\xi}\right) + \sum_{\xi+1}^{\nu} O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{\xi}\right) = O(x).$$

Hence $T_3 = O(1)$ and so $T_2 = O(x^{-1})$.

If $\xi < \mu$ or $\xi \geqslant \nu$, then the estimate for T_1 or T_2 alone suffices.

LEMMA 2. *Theorems* 1 *and* 2 *are true when* $\phi_n = 1$.

Here, and later, we argue as follows. Suppose that

$$\psi(x) = \chi(x)+p(x, c)+q(x, C)+r(x, c, C),$$

where c and C are positive parameters; that η_c and η_C are positive functions of c and C, which tend to 0 when $c \to 0$ or $C \to \infty$; that

$$|p| < \eta_c|\chi| \quad (0 < x \leqslant x_1), \qquad |q| < \eta_C|\chi| \quad (0 < x \leqslant x_2), \tag{2.1}$$

where $x_1 = x_1(c)$ and $x_2 = x_2(C)$; and that

$$r = o(|\chi|) \tag{2.2}$$

when c and C are fixed and $x \to 0$. Then

$$\psi(x) \sim \chi(x). \tag{2.3}$$

For plainly, given any positive ϵ, we have

$$|\psi-\chi| < (\eta_c+\eta_C)|\chi|+|r| < \epsilon|\chi|+|r|$$

for sufficiently small c, sufficiently large C, and $0 < x \leqslant x_3(c, C)$; and so

$$\overline{\lim}\left|\frac{\psi-\chi}{\chi}\right| \leqslant \epsilon.$$

We shall express (2.1) by writing

$$p = O(\eta_c|\chi|), \qquad q = O(\eta_C|\chi|). \tag{2.4}$$

The O's are 'uniform in c or C' in the sense that the constants which they imply are independent of c or C, but not in the sense that the ranges of x for which they hold are independent of c or C.

In the special case relevant here we write

$$\mathbf{S} = \Big(\sum_{n<c/x} + \sum_{c/x\leqslant n\leqslant C/x} + \sum_{n>C/x}\Big) n^{-\alpha}\sin nx = S_1+S_2+S_3. \tag{2.5}$$

Then
$$S_1 = O\Big(x\sum_{n<c/x} n^{1-\beta}\Big) = O(c^{2-\beta}x^{\beta-1}) = O(\eta_c x^{\beta-1}), \tag{2.6}$$

because $|\sin nx| \leqslant nx$; and

$$S_3 = O\{(C/x)^{-\beta}x^{-1}\} = O(C^{-\beta}x^{\beta-1}) = O(\eta_C x^{\beta-1}), \tag{2.7}$$

by partial summation and Lemma 1. Also, if $M = [c/x]$, $N = [C/x]$,

$$S_2 - \int_{c/x}^{C/x} t^{-\alpha}\sin tx\, dt = O(M^{-\beta})+O(N^{-\beta})+ \\ +\sum_{M}^{N-1}\int_{n}^{n+1}\{n^{-\alpha}(\sin nx-\sin tx)+(n^{-\alpha}-t^{-\alpha})\sin tx\}\, dt. \tag{2.8}$$

The first two terms are $o(x^{\beta-1})$, and the last is

$$\sum_M^{N-1} \int_n^{n+1} \{O(xn^{-\beta})+O(n^{-\beta-1})\}\,dt = O\Big(x\sum_M^{N-1} n^{-\beta}\Big)+O(1) = o(x^{\beta-1}).$$

Hence (2.6) gives

$$S_2 = x^{\alpha-1}\int_c^C t^{-\alpha}\sin t\,dt + o(x^{\beta-1}) = x^{\alpha-1}\{J+O(\eta_c)+O(\eta_C)+o(1)\}, \tag{2.9}$$

for fixed c and C. Finally, from (2.5)–(2.9), and our remarks at the beginning of the proof, it follows that $\mathbf{S} \sim x^{\alpha-1}J$. The proof for $\mathbf{C}$ is similar, except that we use $|\cos nx| \leqslant 1$ instead of $|\sin nx| \leqslant nx$ and must suppose $\beta < 1$.

3. We need three more lemmas, in which $t \to \infty$ and $\phi(t)$ is a function of t continuous for $t \geqslant 1$.

LEMMA 3.† *If*

(D′) $\phi(kt) \sim \phi(t)$, *when* $t \to \infty$, *for every fixed positive* k,

then $\phi(kt) \sim \phi(t)$ *uniformly in any finite interval* $0 < k_1 \leqslant k \leqslant k_2$.

Let $$f(t) = \phi(e^t), \qquad \psi = \psi(t,h) = \frac{f(t+h)}{f(t)}.$$

Then $\psi \to 1$ for every fixed h, and we have to show that it does so uniformly in any finite interval of h, say $I(H) = (-H, H)$.

Since $\psi \to 0$ for every h of $I(2H)$, $\psi \to 0$ uniformly in a set K included in $I(2H)$ and of measure greater than $3H$,‡ so that $|\psi| < \epsilon$ for all h of K and $t > t_0(H,\epsilon)$. If h is a given number of $I(H)$, and h_1 and h_2 run through K, then $h-h_1$ runs through a set of measure greater than $3H$ and included in $I(3H)$; and this set and K, having measures whose sum exceeds $6H$, must have points in common; so that every h of $I(H)$ is expressible as $h = h_1+h_2$, where h_1 and h_2 belong to K. Also $|\psi(t,h_1)-1| < \epsilon$ for $t > t_0(H,\epsilon)$ and $|\psi(t+h_1,h_2)-1| < \epsilon$ for $t+h_1 > t_0(H,\epsilon)$; and therefore

$$|\psi(t,h)-1| = |\psi(t,h_1)\psi(t+h_1,h_2)-1| < 2\epsilon+\epsilon^2 < 3\epsilon$$

for $t > t_0(H,\epsilon)+2H$, which proves the lemma.

† We owe this lemma (in which ϕ need not be continuous) to Mr. Besicovitch.

‡ By 'Egoroff's theorem'.

LEMMA 4. *The condition*

$$\text{(A')} \qquad \phi_1(t) = \int_1^t \phi(u)\,du \sim t\phi(t)$$

is equivalent to (D').

(1) If (A') is satisfied, then $\phi_1 \neq 0$ for large t, and

$$\frac{\phi_1'}{\phi_1} = \frac{1}{t} + o\left(\frac{1}{t}\right), \qquad \log\frac{\phi_1(kt)}{\phi_1(t)} = \log k + o(1),\dagger$$

$$\frac{\phi(kt)}{\phi(t)} \sim \frac{1}{k}\frac{\phi_1(kt)}{\phi_1(t)} = \frac{1}{k}e^{\log k + o(1)} \to 1.$$

(2) If (D') is satisfied, $\delta > 0$, l is a positive integer, and $t = 2^l u$, then

$$|\phi(t)| \leqslant 2^{l\delta}|\phi(\tfrac{1}{2}t)| \leqslant \ldots \leqslant 2^{l\delta}|\phi(u)| \tag{3.1}$$

for $u \geqslant u_0(\delta)$. Hence $|\phi(t)| \leqslant \{u^{-l\delta}|\phi(u)|\}t^{l\delta}$, and $t^{-\delta}|\phi(t)| \to 0$. Similarly $t^{\delta}|\phi(t)| \to \infty$ for any positive δ. In particular, $t|\phi(t)| \to \infty$.

In what follows $0 < \epsilon < \frac{1}{2}$ and ζ, $\zeta_1^{(1)}$, $\zeta_2^{(1)}$, $\zeta_2^{(2)}$,... are numbers, depending on various variables, whose moduli are less than ϵ. By Lemma 3, $\phi(u) \sim \phi(t)$ uniformly for $\frac{1}{2}t \leqslant u \leqslant t$. Hence

$$\int_{\frac{1}{2}t}^{t} \phi(u)\,du = \phi(t)\int_{\frac{1}{2}t}^{t}(1+\zeta)\,du = \tfrac{1}{2}t\phi(t)(1+\zeta_1^{(1)}) \quad (t \geqslant t_0(\epsilon)),$$

$$\int_{\frac{1}{4}t}^{\frac{1}{2}t} \phi(u)\,du = \tfrac{1}{4}t\phi(\tfrac{1}{2}t)(1+\zeta_2^{(1)}) = \tfrac{1}{4}t\phi(t)(1+\zeta_2^{(1)})(1+\zeta_2^{(2)}) \quad (\tfrac{1}{2}t \geqslant t_0(\epsilon)),$$

$$\int_{2^{-l}t}^{2^{-l+1}t} \phi(u)\,du = 2^{-l}t\phi(t)(1+\zeta_l^{(1)})(1+\zeta_l^{(2)})\ldots(1+\zeta_l^{(l)}) \quad (2^{-l+1}t \geqslant t_0(\epsilon)).$$

Hence, if $2^{-l}t = \tau$, where $\frac{1}{2}t_0 \leqslant \tau \leqslant t_0$, we have

$$\phi_1(t) - \phi(\tau) = t\phi(t)\sum_{h=1}^{\infty}\frac{(1+\zeta_h^{(1)})\ldots(1+\zeta_h^{(h)})}{2^h},$$

$$\frac{\phi_1(t)}{t\phi(t)} - \sum_1^l \frac{1}{2^h} = \frac{\phi_1(\tau)}{t\phi(t)} + \sum_{h=1}^{l}\frac{(1+\zeta_h^{(1)})\ldots(1+\zeta_h^{(h)})-1}{2^h}. \tag{3.2}$$

The last term has a modulus less than

$$\sum_{h=1}^{\infty}\frac{(1+\epsilon)^h-1}{2^h} < \epsilon\sum_{h=1}^{\infty}h\left(\frac{1+\epsilon}{2}\right)^h = \frac{2\epsilon(1+\epsilon)}{(1-\epsilon)^2} < 12\epsilon.$$

† With the principal values of the logarithms.

Also $l \to \infty$ when $t \to \infty$, and $t|\phi(t)| \to \infty$. Hence it follows from (3.2) that

$$\overline{\lim}\left|\frac{\phi_1(t)}{t\phi(t)}-1\right| < 12\epsilon$$

or $\phi_1 \sim t\phi$.

LEMMA 5. *Either* (A′) *or* (D′) *involves the following consequences*:

(i) $t^{-\delta}|\phi| \to 0$, $t^{\delta}|\phi| \to \infty$, *for any positive* δ;

(ii) *if* $\phi_1^* = \int_1^t |\phi(u)|\,du$, *then* $\phi_1^* \sim t|\phi| \sim |\phi_1|$;

(iii) $\sum_{n\leqslant t} n^{\delta-1}\phi(n) \sim \int_1^t u^{\delta-1}\phi(u)\,du \sim \frac{t^{\delta}}{\delta}\phi(t)$,

for any positive δ, *with similar relations in which* ϕ *is replaced by* $|\phi|$;

(iv) $\sum_{n>t} n^{-\delta-1}\phi(n) \sim \int_t^{\infty} u^{-\delta-1}\phi(u)\,du \sim \frac{t^{-\delta}}{\delta}\phi(t)$,

with a similar gloss.†

Of these assertions, (i) has been proved already. For (ii), $|\phi(kt)| \sim |\phi(t)|$ and so $\phi_1^* \sim t|\phi| \sim |\phi_1|$. To prove (iii) we have

$$\begin{aligned} \int_1^t u^{\delta-1}\phi\,du &= t^{\delta-1}\phi_1+(1-\delta)\int_1^t u^{\delta-2}\phi_1\,du, \\ \int_1^t u^{\delta-1}|\phi|\,du &= t^{\delta-1}\phi_1^*+(1-\delta)\int_1^t u^{\delta-2}\phi_1^*\,du. \end{aligned} \tag{3.3}$$

Since $\phi_1^* \sim t|\phi|$ and $t^{\delta-1}\phi_1^* \to \infty$, the second of (3.3) gives

$$\delta\int_1^t u^{\delta-1}|\phi|\,du \sim t^{\delta-1}\phi_1^* \sim t^{\delta}|\phi|.$$

But then

$$\int_1^t u^{\delta-2}\phi_1\,du = \int_1^t u^{\delta-1}\phi\,du + o\left(\int_1^t u^{\delta-1}|\phi|\,du\right) = \int_1^t u^{\delta-1}\phi\,du + o(t^{\delta}|\phi|),$$

and the first of (3.3) gives $\delta\int_1^t u^{\delta-1}\phi\,du \sim t^{\delta-1}\phi_1 \sim t^{\delta}\phi$.

† We prove a little more than we actually need, for the sake of symmetry.

We have thus proved the assertions about integrals in (iii). Next

$$\left|\sum_{n\leqslant t} n^{\delta-1}\phi(n) - \int_1^t u^{\delta-1}\phi\,du\right| \leqslant \sum_{n=1}^{[t]} \int_n^{n+1} |n^{\delta-1}\phi(n) - u^{\delta-1}\phi(u)|\,du + \left|\int_t^{[t]+1} u^{\delta-1}\phi(u)\,du\right|.$$

Here $n^{\delta-1}-u^{\delta-1} = O(n^{\delta-2})$, $\phi(u) \sim \phi(n)$, and the last integral is plainly $o(t^\delta|\phi|)$. Hence the sum is

$$o\left(\sum_{n=1}^{[t]} \int_n^{n+1} u^{\delta-1}|\phi|\,du\right) = o\left(\int_1^{t+1} u^{\delta-1}|\phi|\,du\right) = o(t^\delta|\phi|),$$

and

$$\sum_{n\leqslant t} n^{\delta-1}\phi(n) \sim \int_1^t u^{\delta-1}\phi\,du \sim \frac{t^\delta}{\delta}\phi(t).$$

The sum with $|\phi(n)|$ may be dealt with similarly, and this completes the proof of (iii). That of (iv) follows the same lines and need not be set out in detail.

4. We can now prove Theorems 1 and 2. It is convenient for technical reasons to suppose that b_n and ϕ_n are the values, for $t = n$, of continuous functions $b(t)$ and $\phi(t)$, the conditions on these functions corresponding to (A)–(C) of the theorems being (A′) $\phi_1 \sim t\phi$; (B 1′) ϕ is real and $|b|$ decreases; (B 2′) ϕ is real and increasing; (B 3′) $\phi' = O(t^{-1}|\phi|)$; (C 1′) ϕ is real, $t^\delta\phi$ increases, $t^{-\delta}\phi$ decreases; (C 2′) $\phi' = o(t^{-1}|\phi|)$. These conditions are in fact satisfied by the $b(t)$ or $\phi(t)$ obtained from b_n or ϕ_n by linear interpolation.†

After Lemma 4 we may replace (A′) by $\phi(kt) \sim \phi(t)$, or (A) by $\phi_{[kn]} \sim \phi_n$.

We decompose **S** (assuming its convergence provisionally) as in (2.5), and we have to show that

$$S_1 = O\left\{\frac{\eta_c}{x}\left|b\left(\frac{1}{x}\right)\right|\right\}, \tag{4.1}$$

$$S_3 = O\left\{\frac{\eta_C}{x}\left|b\left(\frac{1}{x}\right)\right|\right\}, \tag{4.2}$$

$$S_2 = \{J + O(\eta_c) + O(\eta_C) + o(1)\}\frac{1}{x}b\left(\frac{1}{x}\right). \tag{4.3}$$

† We must interpolate b_n linearly in case (B 1), ϕ_n in case (B 2) or (C 1): otherwise either interpolation is effective. In either case $b(t)$ or $\phi(t)$ will have angles at the integers, but we may interpret (B 3′) or (C 2′) as referring to the backward and forward derivatives (or 'smooth out' the angles if we prefer to).

First,

$$S_1 = \sum_{n<c/x} n^{-\alpha}\phi(n)\sin nx = O\Big\{x \sum_{n<c/x} n^{1-\beta}|\phi(n)|\Big\} = O\Big\{c^{2-\beta}x^{\beta-1}\Big|\phi\Big(\frac{c}{x}\Big)\Big|\Big\},$$

by (iii) of Lemma 5. Since $\phi(c/x) \sim \phi(1/x)$, this gives (4.1).†

Next

$$S_2 = \sum_{c/x}^{C/x} n^{-\alpha}\phi(n)\sin nx = \phi\Big(\frac{1}{x}\Big)\sum_{c/x}^{C/x} n^{-\alpha}\sin nx + o\Big\{\Big|\phi\Big(\frac{1}{x}\Big)\Big|\sum_{c/x}^{C/x} n^{-\beta}\Big\}.$$

The last term is $o\{x^{\beta-1}|\phi(1/x)|\}$; and so (4.3) follows from (2.9).

So far we have used only hypothesis (A′). In discussing S_3 we must assume one of (B 1′), (B 2′), and (B 3′).

(*a*) If (A′) and (B 1′) are satisfied then, after Lemma 5 (i), $|b_n| \to 0$, and S_3 is certainly convergent. And, if $N = [C/x]$, then

$$S_3 = \sum_{n>C/x} b_n \sin nx = b_{N+1}\, O\Big(\frac{1}{x}\Big) = O\Big\{\frac{1}{x}b\Big(\frac{C}{x}\Big)\Big\}$$
$$= O\Big\{\frac{C^{-\beta}}{x}b\Big(\frac{1}{x}\Big)\Big\} = O\Big\{\frac{\eta_C}{x}b\Big(\frac{1}{x}\Big)\Big\}.$$

(*b*) The convergence of S_3 is less obvious when (B 1′) is replaced by (B 2′). We then write

$$s_l = \sum_{2^lC/x<n\leqslant 2^{l+1}C/x} n^{-\beta-i\gamma}\phi(n)\sin nx \quad (l = 0, 1, 2,\ldots).$$

Summing partially,‡ and using Lemma 1 and (3.1), we see that

$$s_l = O\Big\{\Big(\frac{2^lC}{x}\Big)^{-\beta}\phi\Big(\frac{2^{l+1}C}{x}\Big)\frac{1}{x}\Big\} = O\Big\{C^{-\beta}2^{l(\delta-\beta)}x^{\beta-1}\phi\Big(\frac{1}{x}\Big)\Big\}$$

for any positive δ; and the same estimate is valid for any partial sum of s_l. We may take $\delta = \frac{1}{2}\beta$, and then it becomes plain that $\sum s_l$, S_3, and $\mathbf{S}$ are convergent, and that S_3 is of the form (4.2).

(*c*) Finally, suppose that (B 3′) is satisfied; that $N = [C/x]$; and that

$$\chi_n = \sum_n^\infty m^{-\alpha}\sin mx,$$

so that $\chi_n = O(n^{-\beta}x^{-1})$, after Lemma 1. Then

$$\sum_N^{N'} b_n \sin nx = \sum_N^{N'} \phi_n(\chi_n - \chi_{n+1}) = \phi_N\chi_N - \phi_{N'}\chi_{N'+1} - \sum_N^{N'-1}\chi_{n+1}\Delta\phi_n.$$

† $|S_1| < Hc^{2-\beta}x^{\beta-1}|\phi(c/x)|$, where H is independent of c, for $0 < x < x_0(c)$, and $\phi(c/x) \sim \phi(1/x)$ for every c, imply $|S_1| < 2Hc^{2-\beta}x^{\beta-1}|\phi(1/x)|$ for $0 < x < x_1(c)$. We shall argue thus repeatedly.

‡ Twice, once on $n^{-\alpha}\sin nx$ and once on $n^{-i\gamma}\sin nx$.

Now $\phi_n \chi_n = O(n^{\delta-\beta}x^{-1})$, for any positive δ, and $\Delta\phi_n = O(n^{-1}|\phi_n|)$, by (B 3′). It follows that S_3 is convergent, and that

$$S_3 = \phi_N \chi_N - \sum_N^\infty \chi_{n+1}\Delta\phi_n = O\left(\frac{N^{-\beta}}{x}|\phi_N|\right) + O\left(\frac{1}{x}\sum_N^\infty n^{-1-\beta}|\phi_n|\right).$$

The second term here may be absorbed in the first, by (iv) of Lemma 5; and so

$$S_3 = O\left\{C^{-\beta}x^{\beta-1}\left|\phi\left(\frac{C}{x}\right)\right|\right\} = O\left\{\frac{\eta_C}{x}\left|b\left(\frac{1}{x}\right)\right|\right\}.$$

5. This completes the proof of Theorem 1, so far as hypotheses (A) and (B) are concerned. As regards hypotheses (C) or (C′), it is plain that either of them implies one of (B) or (B′). It is therefore sufficient to prove that either (C 1′) or (C 2′) implies (A′).

(*a*) If (C 1′) is satisfied, then

$$\phi_1 = \int_1^t u^\delta \phi . u^{-\delta}\, du \leqslant t^\delta \phi \frac{t^{1-\delta}}{1-\delta} = \frac{t\phi}{1-\delta}$$

for every positive δ. Hence

$$\overline{\lim}\frac{\phi_1}{t\phi} \leqslant \frac{1}{1-\delta}, \qquad \overline{\lim}\frac{\phi_1}{t\phi} \leqslant 1;$$

and similarly the lower limit is at least 1.

(*b*) If (C 2′) is satisfied, then

$$\phi_1 = t\phi - \int_1^t u\phi'\, du = t\phi + o\left(\int_1^t |\phi|\, du\right) = t\phi + o(\phi_1^*),$$

$$\phi_1^* = t|\phi| - \int_1^t u|\phi|'\, du = t|\phi| + o(\phi_1^*),$$

since $|\phi|' \leqslant |\phi'|$. It follows by comparison that $\phi_1^* \sim t|\phi|$ and $\phi_1 \sim t\phi$.

This completes the proof of Theorem 1. The proof of Theorem 2 is the same except for the discussion of S_1. Here

$$S_1 = \sum_{n<c/x} n^{-\alpha}\phi(n)\cos nx = O\left\{\sum_{n<c/x} n^{-\beta}|\phi(n)|\right\} = O\left\{c^{1-\beta}x^{\beta-1}\left|\phi\left(\frac{1}{x}\right)\right|\right\}:$$

we must suppose $\beta < 1$.† A typical case of either theorem is that in which a_n or b_n is of the form $n^{-\alpha}\psi_n e^{i\chi_n}$, where $\alpha = \beta + i\gamma$, $0 < \beta < 1$, and ψ_n and χ_n are '*L*-functions', in the sense of Hardy's

† $\delta > 0$ in (iii) of Lemma 5.

Orders of infinity, of the ranges $n^{-\delta} \prec \psi_n \prec n^{\delta}$ and $1 \prec \chi_n \prec \log n$. Thus a_n or b_n might be

$$n^{-\alpha}(\log n)^{A}(\log\log n)^{B}e^{C i(\log n)^{\rho}} \quad (0 < \rho < 1).$$

6. There are naturally similar theorems for integrals

$$\int_0^{\infty} a(t)\cos xt\, dt, \qquad \int_0^{\infty} b(t)\sin xt\, dt,$$

which the reader can formulate for himself. Here, however, there is a dual problem, since x may tend to ∞ instead of 0, in which case it is the behaviour of $a(t)$ or $b(t)$ for *small* t that is important. This case corresponds to the problem of finding asymptotic formulae for the Fourier constants of a function with a singularity of assigned nature at the origin.

There are a good many known theorems which are special cases of Theorems 1 and 2 or their integral analogues. Different writers have tended to consider slightly different problems, and the literature is fragmentary. We may refer to Bromwich, *Infinite Series*, 494; Hardy, *Proc. London Math. Soc.* (2), 32 (1931), 441–8; Haslam-Jones, *Journal London Math. Soc.* 2 (1927), 151–4; Titchmarsh, ibid., 1 (1926), 35–7, and *Fourier Integrals*, 172–4. Theorems with wider conditions but less precise results have been proved by Salem, *Comptes rendus*, 186 (1928), 1804–6, and Young, *Proc. London Math. Soc.* (2), 12 (1913), 433–52: for these see Zygmund, *Trigonometrical Series*, 112–16. The theorems proved by Hardy, *Messenger of Math.* 58 (1929), 130–5, and Young, *Proc. Royal Soc.* (A), 93 (1917), 42–55, are of a rather different character.

[*Added Feb. 1944.*] Lemma 1 is a case of the following more general theorem, in which

$$A_n = a_0+a_1+\ldots+a_n, \qquad s_n(z) = a_0+a_1 z+\ldots+a_n z^n, \qquad |z| \leqslant 1,$$

H is an absolute constant, and $H(\beta)$ a function of β only.

If $0 < \beta < 1$, $a_n = o(1)$, $\Delta a_n = O(n^{\beta-2})$, *then*

$$|s_n(z)| < H(\beta)|1-z|^{-\beta}.$$

The conditions are not sufficient when $\beta = 0$ *or* $\beta = 1$. *But, if* $A_n = O(1)$, $\Delta a_n = O(n^{-2})$, *then* $|s_n(z)| < H$; *and if* $a_n = O(1)$, $\Delta^2 a_n = O(n^{-2})$, *then* $|s_n(z)| < H|1-z|^{-1}$.

This theorem includes (besides Lemma 1) a number of more special theorems proved by Hardy, *Quarterly Journal*, 44 (1917), 147–60; Landau, *Ergebnisse der Funktionentheorie* (1929), 68–9; M. Riesz, *Acta Univ. Hungaricae*, 1 (1923), 104–13; Szegö, *Math. Zeitschrift*, 25 (1926), 172–87.

GAUSS SUMS AND PRIMITIVE CHARACTERS

By A. P. GUINAND (*Oxford*)

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1. Introduction

The reciprocity formula for Gauss sums may be written in the form

$$\sum_{n=1}^{q} \exp(n^2\pi ip/q) = e^{\frac{1}{4}\pi i}\left(\frac{q}{p}\right)^{\frac{1}{2}} \sum_{n=1}^{p} \exp(-n^2\pi iq/p) \qquad (1.1)$$

where p and q are positive integers.*

In this paper it is shown that reciprocity formulae also hold for similar sums involving primitive characters as coefficients. For even primitive characters† a formula directly analogous to (1.1) holds, but for odd primitive characters the direct analogy breaks down, and a more complicated relation between sums holds instead.

The method used is based on Dirichlet's proof‡ of (1.1).

2. Preliminary lemma

LEMMA.§ *If $\chi(n)$ is a primitive character modulo k ($k > 1$), and $F(x)$ is continuous and of bounded variation in the finite interval (a, b), then*

$$\sum_{a\leqslant n\leqslant b}{}' \chi(n)F(n) = \frac{\tau(\chi)}{k} \lim_{N\to\infty} \sum_{n=-N}^{N} \bar{\chi}(n) \int_a^b e^{-2\pi int/k} F(t)\, dt,$$

where the dash indicates that the terms $n = a$ or $n = b$ are to be halved if a or b is an integer, $\bar{\chi}(n)$ is the conjugate of $\chi(n)$, and

$$\tau(\chi) = \sum_{n=1}^{k} \chi(n) e^{2\pi in/k}.$$

* E. Lindelöf, *Calcul des résidus* (Paris, 1905), 73, and G. H. Hardy and J. E. Littlewood, *Proc. Fifth International Congress of Mathematicians* (1912), 223–9.

† The usual notation for characters is used. Cf. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen, I* (Leipzig, 1909), 486–94. We call $\chi(n)$ an even or an odd character according as $\chi(-n) = \pm\chi(n)$.

‡ L. Dirichlet, *Werke*, 1 (1889), 473–9.

§ This is a simple case of a formula given by L. J. Mordell, *Proc. Cambridge Phil. Soc.* 24 (1928), 585–96. Corresponding real forms of an extension of Poisson's summation formula are given by A. P. Guinand, *Annals of Math.* 42 (1941), 591–603.

3. Reciprocity formula for sums with even characters

If $\chi(n)$ is an even primitive character modulo k,

$$\sum_{n=-qk}^{qk}\chi(n)\exp(n^2\pi ip/qk) = 2\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk). \tag{3.1}$$

The terms $n = \pm qk$ vanish since $\chi(\pm qk) = 0$; hence, by the lemma, this sum is equal to

$$\frac{\tau(\chi)}{k}\lim_{N\to\infty}\sum_{n=-N}^{N}\bar\chi(n)\int_{-qk}^{qk}\exp\left(\frac{\pi ip}{qk}t^2-\frac{2\pi int}{k}\right)dt.$$

Putting $u = \frac{p}{q}t-n$ and $n = 2pkr+m$ this becomes

$$q\frac{\tau(\chi)}{pk}\sum_{m=-pk}^{pk-1}\bar\chi(m)\exp\left(\frac{-m^2\pi iq}{pk}\right)\lim_{R\to\infty}\sum_{r=-R}^{R}\int_{(2r-1)pk+m}^{(2r+1)pk+m}\exp\left(\frac{\pi iu^2q}{pk}\right)du. \tag{3.2}$$

The limit of the inner sum is

$$\int_{-\infty}^{\infty}\exp(\pi iu^2q/pk)\,du = e^{\frac{1}{4}\pi i}\left(\frac{pk}{q}\right)^{\frac{1}{2}}.$$

Hence (3.2) is equal to

$$e^{\frac{1}{4}\pi i}\left(\frac{q}{pk}\right)^{\frac{1}{2}}\tau(\chi)\sum_{m=-pk}^{pk-1}\bar\chi(m)\exp(-m^2\pi iq/pk).$$

Treating this sum as in (3.1) we have the theorem:

THEOREM 1. *If $\chi(n)$ is an even primitive character modulo k, and p and q are positive integers, then*

$$\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk) = e^{\frac{1}{4}\pi i}\left(\frac{q}{pk}\right)^{\frac{1}{2}}\tau(\chi)\sum_{n=1}^{pk}\bar\chi(n)\exp(-n^2\pi iq/pk).$$

4. Reciprocity formula for sums with odd characters

If $\chi(n)$ is an odd primitive character, the sum analogous* to (3.1) is

$$\sum_{n=-qk}^{qk}n\chi(n)\exp(n^2\pi ip/qk) = 2\sum_{n=1}^{qk}n\chi(n)\exp(n^2\pi ip/qk). \tag{4.1}$$

As in § 3 this sum is equal to

* By analogy with functional equations for theta functions involving odd characters. Cf. E. Landau, loc. cit. 486–94.

$$\frac{\tau(\chi)}{k}\lim_{N\to\infty}\sum_{n=-N}^{N}\bar{\chi}(n)\int_{-qk}^{qk} t\exp\left(\frac{\pi ip}{qk}t^2-\frac{2\pi int}{k}\right)dt$$

$$=\left(\frac{q}{p}\right)^2\frac{\tau(\chi)}{k}\sum_{m=-pk}^{pk}\bar{\chi}(m)\exp(-m^2\pi iq/pk)\times$$

$$\times\lim_{R\to\infty}\sum_{r=-R}^{R}\int_{(2r-1)pk+m}^{(2r+1)pk+m}(2pkr+m-u)\exp(\pi iu^2q/pk)\,du$$

$$\equiv\left(\frac{q}{p}\right)^2\frac{\tau(\chi)}{k}(S_1+S_2-S_3),\text{ say,} \tag{4.2}$$

where S_1, S_2, S_3 arise respectively from the three terms of the first factor of the integrand. By proceeding as in § 3, S_2 leads to a term

$$e^{\frac{1}{4}\pi i}k^{-\frac{1}{2}}\tau(\chi)\left(\frac{q}{p}\right)^{\frac{3}{2}}\sum_{n=-pk}^{pk}n\bar{\chi}(n)\exp(-n^2\pi iq/pk). \tag{4.3}$$

Further, the integral in S_3 is

$$\int_{(2r-1)pk+m}^{(2r+1)pk+m}u\exp(\pi iu^2q/pk)\,du=0,$$

and so S_3 vanishes. From (4.2),

$$S_1=2pk\sum_{m=-pk}^{pk}\bar{\chi}(m)\exp(-m^2\pi iq/pk)\lim_{R\to\infty}T_m(R) \tag{4.4}$$

where $$T_m(R)=\sum_{r=-R}^{R}r\int_{(2r-1)pk+m}^{(2r+1)pk+m}\exp(\pi iu^2q/pk)\,du.$$

Rearranging the sums and integrals, and putting $u=(2r+1)pk+v$ for r positive and $u=(-2r+1)pk-v$ for r negative, we obtain

$$T_m(R)=(-)^{pqk}2i\sum_{r=1}^{R}r\int_{-m}^{m}\exp\left(\frac{\pi iq}{pk}v^2+4\pi iqrv\right)\sin 2\pi qv\,dv. \tag{4.5}$$

Summing the series in r this becomes

$$T_m(R)=(-)^{pqk}\{U_m(R)-V_m(R)\}, \tag{4.6}$$

where

$$U_m(R)=R\int_{-m}^{m}\exp\left\{\frac{\pi iq}{pk}v^2+2\pi iqv(2R+1)\right\}dv,$$

$$V_m(R)=\int_{-m}^{m}\exp\left\{\frac{\pi iq}{pk}v^2+2\pi iqvR\right\}\frac{\sin 2\pi qvR}{\sin 2\pi qv}\,dv. \tag{4.7}$$

Integrating $U_m(R)$ by parts, the integrated terms vanish, giving

$$U_m(R) = -\frac{R}{pk(2R+1)} \int_{-m}^{m} v \exp\left\{\frac{\pi iq}{pk} v^2 + 2\pi iqv(2R+1)\right\} dv,$$

and by the Riemann-Lebesgue theorem* this tends to zero as $R \to \infty$.

Putting $t = \pm 2qv$, according as $v \gtrless 0$, in (4.7)

$$V_m(R) = \frac{1}{2q} \int_{0}^{2mq} \exp\left(\frac{\pi it^2}{4pqk}\right) \frac{\sin 2\pi Rt}{\sin \pi t}\, dt. \tag{4.8}$$

By Fourier's single-integral theorem†

$$\lim_{R\to\infty} \int_{n}^{n+\frac{1}{2}} \exp\left(\frac{\pi it^2}{4pqk}\right) \frac{\sin 2\pi Rt}{\sin \pi t}\, dt = (-)^n \tfrac{1}{2} \exp\left(\frac{\pi in^2}{4pqk}\right),$$

and similarly for the integral over the range $(n-\frac{1}{2}, n)$. Splitting the integral (4.8) up into a series of such ranges, we have

$$\begin{aligned} \lim_{R\to\infty} V_m(R) &= \frac{1}{2q} \sum_{n=0}^{2mq}{}' (-)^n \exp\left(\frac{\pi in^2}{4pqk}\right) \\ &= \frac{1}{4q} \sum_{n=-2mq}^{2mq}{}' (-)^n \exp\left(\frac{\pi in^2}{4pqk}\right). \end{aligned}$$

Now, by (4.5), $T_{-m}(R) = -T_m(R)$. Hence, by (4.4) and (4.6),

$$\begin{aligned} S_1 &= 4pk \sum_{m=1}^{pk} \bar{\chi}(m) \exp(-m^2\pi iq/pk) \lim_{R\to\infty} T_m(R) \\ &= (-)^{pqk+1} \frac{pk}{q} \sum_{m=1}^{pk} \bar{\chi}(m) \exp(-m^2\pi iq/pk) \sum_{n=-2mq}^{2mq}{}' (-)^n \exp\left(\frac{\pi in^2}{4pqk}\right). \end{aligned}$$

The terms $n = \pm 2mq$ give rise to sums

$$\sum_{m=1}^{pk} \bar{\chi}(m) = 0,$$

so, for simplicity of notation, we need not halve these terms. Hence

$$S_1 = (-)^{pqk+1} \frac{pk}{q} \sum_{m=1}^{pk} \sum_{n=-2mq}^{2mq} (-)^n \bar{\chi}(m) \exp\left\{\frac{\pi i(n^2 - 4m^2q^2)}{4pqk}\right\}. \tag{4.9}$$

By (4.2), (4.3), (4.9) we have the theorem:

* E. C. Titchmarsh, *Theory of Functions* (Oxford, 1939), 403.

† E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 25.

THEOREM 2. *If $\chi(n)$ is an odd primitive character modulo k, and p and q are positive integers, then*

$$\sum_{n=1}^{qk} n\chi(n)\exp(n^2\pi ip/qk) - e^{\frac{1}{4}\pi i}k^{-\frac{1}{2}}\tau(\chi)\left(\frac{q}{p}\right)^{\frac{3}{2}}\sum_{n=1}^{pk} n\bar{\chi}(n)\exp(-n^2\pi iq/pk)$$
$$= (-)^{pqk+1}\frac{q}{2p}\tau(\chi)\sum_{m=1}^{pk}\sum_{n=-2mq}^{2mq}(-)^n\bar{\chi}(m)\exp\left\{\frac{\pi i(n^2-4m^2q^2)}{4pqk}\right\}.$$

5. Elementary properties of the sums

Some further properties of these sums can be proved by simple substitutions. For instance, putting $n = qk-m$ we find that

$$\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk) = (-)^{pqk}\chi(-1)\sum_{m=1}^{qk}\chi(m)\exp(m^2\pi ip/qk). \quad (5.1)$$

Hence this sum vanishes if $(-)^{pqk}\chi(-1) = -1$. In particular the sums in (1.1) vanish if p and q are both odd.

From (5.1) and similar results we can prove the theorem:

THEOREM 3. *If $\chi(n)$ is a character modulo k, and p, q, r, k are positive integers, then*

(i) *if $\chi(n)$ is even and p, q, k are all odd,*

$$\sum_{n=1}^{rqk}\chi(n)\exp(n^2\pi ip/qk) = 0;$$

(ii) *if $\chi(n)$ is even and p, q, k are not all odd,*

$$\sum_{n=1}^{rqk}\chi(n)\exp(n^2\pi ip/qk) = r\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk);$$

(iii) *if $\chi(n)$ is odd and p, q, k are all odd,*

$$\sum_{n=1}^{qk} n\chi(n)\exp(n^2\pi ip/qk) = \tfrac{1}{2}qk\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk),$$
$$\sum_{n=1}^{rqk} n\chi(n)\exp(n^2\pi ip/qk) = (-)^{r+1}r\sum_{n=1}^{qk} n\chi(n)\exp(n^2\pi ip/qk);$$

(iv) *if $\chi(n)$ is odd and p, q, k are not all odd,*

$$\sum_{n=1}^{qk}\chi(n)\exp(n^2\pi ip/qk) = 0,$$
$$\sum_{n=1}^{rqk} n\chi(n)\exp(n^2\pi ip/qk) = r\sum_{n=1}^{qk} n\chi(n)\exp(n^2\pi ip/qk).$$

A THEOREM ON HAUSDORFF'S METHODS OF SUMMATION

By W. H. J. FUCHS (*Swansea*)

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1. Introduction. A sequence $\{s_n\}$ is said to be 'summable by the Hausdorff method T with moments μ_n $(T \sim \mu_n)$', if

$$t_n = \sum_{k=0}^{n} \binom{n}{k} s_k \Delta^{n-k}\mu_k = \sum_{k=0}^{n} \binom{n}{k} s_k \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \mu_{k+p}$$

tends to a limit as n tends to infinity.† The product of $T \sim \mu_n$ and $S \sim \mu_n'$ is again a Hausdorff method and $ST = TS \sim \mu_n \mu_n'$. I say that the method T_2 *is at least as strong as the method* T_1 $(T_2 \supseteq T_1)$, if every sequence of real numbers summed by T_1 is also summed by T_2. A method is *regular*, if it is at least as strong as ordinary convergence.

Generalizing results of Rogosinski I shall prove

THEOREM 1. *Let* $T_1 \sim \mu_n^{(1)}$ *and* $T_2 \sim \mu_n^{(2)}$ *be regular Hausdorff methods. Then* $T_2 \supseteq T_1$, *if and only if* $T_2 = \Theta T_1$, *where* Θ *is a regular Hausdorff method.*

The sufficiency part of the theorem is trivial. The necessity of the condition has been proved by Rogosinski for the case when $\mu_n^{(1)} \neq 0$ with at most a finite number of exceptions.‡ For 'reasonable' methods all the moments are different from zero, and in this case the result has been known for some time.§ Even so Theorem 1 may be of some interest because it gives great logical simplicity to the theory of strength of regular Hausdorff methods.

It is well known that the method $T \sim \mu_n$ is regular if and only if the μ_n are moment constants of a function of bounded variation (b.v.),‖

$$\mu_n = \int_0^1 t^n \, d\psi(t).$$

We write

$$T(z) = \int_0^1 t^z \, d\psi(t) \tag{1.1}$$

† For the theory of Hausdorff methods see (**5**) and (**6**). I follow the notation of (**6**). All methods of summation occurring in this paper are Hausdorff methods.

‡ (**6**), 171, Theorem 2. § See e.g. (**9**), 122. ‖ (**5**), 98.

for the 'Mellin transform' (M.t.) associated with T. Then $\mu_n = T(n)$. We can state Theorem 1 in the form $T_2 \supseteq T_1$ if and only if

$$T_2(n) = \Theta(n)T_1(n) \quad (n = 0, 1, 2, \ldots).$$

By Carlson's theorem it follows that

$$T_2(z) = \Theta(z)T_1(z).$$

This leads to the following formulation of Theorem 1. *If T_1 and T_2 are regular, then $T_2 \supseteq T_1$ if and only if $T_2(z)/T_1(z)$ is a Mellin transform.*

The function (1.1) is regular and bounded in $\Re z > 0$. If it has zeros at the points $b_1, b_2, \ldots$ of the positive real axis, then $\sum b_\nu^{-1} < \infty$ by Carleman's formula.† Theorem 1 is therefore a consequence of

THEOREM 2. *Let $T_1 \sim \mu_n^{(1)}$, $T_2 \sim \mu_n^{(2)}$. If $\mu_n^{(1)} \neq 0$ except for $n = a_1, a_2, \ldots$ $(1 \leqslant a_1 < a_2 < \ldots)$‡ where $\sum a_\nu^{-1} < \infty$, then $T_2 \supseteq T_1$ implies that $T_2 = \Theta T_1$, where Θ is regular.*

We exclude here and in the sequel the trivial case $\mu_n = 0$ $(n \geqslant 1)$; when $\mu_n^{(1)} = 0$ $(n \geqslant 1)$, every sequence is summed by T_1. It is easily seen that Theorem 1 is true in this case.

Theorem 2 proves a little more than necessary, since T_1 and T_2 no longer need to be regular.

It can be shown that the conclusion of Theorem 2 ceases to hold, if $\sum a_\nu^{-1}$ diverges too rapidly.§ The question remains open whether the condition on the a_ν of Theorem 2 is best possible.

Theorem 1 (and *a fortiori* Theorem 2) is false for 'continuous Hausdorff methods' defined by transformations of the type

$$t(x) = \int_0^1 s(xt)\, d\psi(t).$$

A *Gegenbeispiel* is provided by the 'Euler methods' E_α $(1 \leqslant \alpha)$ for which $t(x) = s(x/\alpha)$. All these methods are obviously equivalent to ordinary convergence E_1. But it is easily verified that the quotient of the M.t. is

$$E_1(z)/E_\alpha(z) = \alpha^z.$$

This function is unbounded in the right half-plane, so that it cannot be an M.t. Therefore Theorem 1 is not true in this case.‖

† (**2**).

‡ A change in the value of $\mu_0^{(1)}$ does not affect the strength of T_1 ((**6**), 170). We may therefore assume $\mu_0^{(1)} \neq 0$.

§ (**4**), 196.

‖ This *Gegenbeispiel* is taken from (**7**), 345.

2. Notation and Lemmas. In this section I summarize the main abbreviations and state a few lemmas.

Integrals with unspecified limits are to be taken from 0 to 1. All numbers are real, unless otherwise stated.

We write V for the normed, linear space of functions $\psi(t)$ of b.v. $(0 \leqslant t \leqslant 1)$ with $\psi(1) = 0$. The norm in V is defined by $\|\psi\| = \int |d\psi|$. By G we denote the sub-space of absolutely continuous (a.c.) functions $g(t)$ of V. The functions of V satisfying†

$$\int t^{a_\nu}\, d\psi(t) = 0 \quad (\nu = 1, 2, \ldots)$$

form the sub-space V^*, and G^* is the space of all a.c. functions in V^*.

The first two lemmas are well-known theorems on linear operations.

LEMMA 1. (i) *Let $h_1(t), h_2(t), \ldots$ be integrable functions. The equations*

$$\int x(t) h_k(t)\, dt = \alpha_k \quad (k = 1, 2, \ldots)$$

can be satisfied by a measurable function $x(t)$ with essential upper bound $|x(t)| \leqslant K$, if and only if, for every finite set of λ,

$$|\sum \lambda_k \alpha_k| \leqslant K \int |\sum \lambda_k h_k(t)|\, dt.$$

(ii) *The equations*

$$\int t^k\, d\psi(t) = \beta_k \quad (k = 0, 1, 2, \ldots)$$

can be satisfied by a function $\psi(t)$ with $\int |d\psi| \leqslant K$ if and only if, for every finite set of λ,

$$|\sum \lambda_k \beta_k| \leqslant K \max_{0 \leqslant t \leqslant 1} |\sum \lambda_k t^k|.‡$$

LEMMA 2. *If $\alpha(t)$ is measurable, and the essential upper bound $|\alpha|$ is M, then*

$$A(g) = \int \alpha(t) g'(t)\, dt \tag{2.1}$$

defines a linear functional in the space G. The norm of this functional is M, i.e.

$$\text{up. bd.}_{g \in G;\, g \neq 0} |A(g)|\, \|g\|^{-1} = M. \tag{2.2}$$

Every linear functional defined for all g of G and satisfying (2.2) *can be represented in the form* (2.1).

† The a_ν are the same as in Theorem 2.

‡ For a proof see (1), 74–5.

The transformation $g \to g'$ defines an isomorphism between G and $L(0,1)$. For the latter space the result is well known.†

The next lemma is closely related to the results of (3) and essentially contained in them. I give the proof (found without any knowledge of (3)), because I require the estimate (2.3).

LEMMA 3. *If the linear functional* (2.1) *vanishes for every* g *of* G^*, *then*

$$\alpha(t) = \sum_1^\infty \Lambda_\nu t^{a_\nu} \quad \text{(p.p.)}.$$

Further, $$|\Lambda_\nu| < MC_\nu,$$

where C_ν *depends on the* a_ν *only and*

$$\lim_{\nu\to\infty} C_\nu^{1/a_\nu} = 1. \tag{2.3}$$

Proof. We observe first that

$$\text{l. bd.} \int |\alpha(t) + \sum \lambda_\nu t^{a_\nu}|\, dt = 0,$$

as the λ vary through all finite sets of values. For otherwise

$$c \leqslant \int |\alpha(t) + \sum \lambda_\nu t^{a_\nu}|\, dt$$

for some positive c and all finite sets of the λ. But, by Lemma 1 (i), this implies that the equations

$$\int \alpha(t) g'(t)\, dt = c, \qquad \int t^{a_\nu} g'(t)\, dt = 0 \quad (\nu = 1, 2, \ldots)$$

have an integrable solution $g'(t)$. This contradicts our hypothesis. We can therefore find a sequence of polynomials $P_n(t) = \sum \lambda_\nu^{(n)} t^{a_\nu}$ such that

$$\int |\alpha(t) - P_n(t)|\, dt < Mn^{-1} \quad (n = 1, 2, \ldots).$$

Then

$$|Q_n(t)| = \left| \sum \frac{\lambda_\nu^{(n)} t^{a_\nu+1}}{a_\nu+1} \right| = \left| \int_0^t P_n(u)\, du \right|$$

$$\leqslant \int_0^t |\alpha - P_n|\, du + \int_0^t |\alpha|\, du < 2M,$$

and so $\int |Q_{1n}|^2\, dt < 4M^2$. It follows immediately from calculations given in (8) that

$$|\lambda_\nu^{(n)}| < MC_\nu, \tag{2.4}$$

where

$$C_\nu = 2\sqrt{2}(a_\nu+2)^{\frac{1}{2}} \prod_{\mu\neq\nu} \left| \frac{a_\mu + a_\nu + 3}{a_\mu - a_\nu} \right|.$$

† (1), 61–5.

In the same way we deduce from

$$|Q_n-Q_m| \leqslant \int |\alpha-P_n|\,dt + \int |\alpha-P_m|\,dt < M(n^{-1}+m^{-1})$$

that $$|\lambda_\nu^{(n)}-\lambda_\nu^{(m)}| < \tfrac{1}{2}M(n^{-1}+m^{-1})C_\nu.$$

By Cauchy's test $$\Lambda_\nu = \lim_{n\to\infty} \lambda_\nu^{(n)} \tag{2.5}$$

exists for every ν.

Next we prove (2.3).

$$C_\nu = 2\sqrt{2}(a_\nu+2)^{\frac{1}{2}} \prod_{\mu<\nu} \times \prod_{a_\nu<a_\mu<2a_\nu} \times \prod_{a_\mu\geqslant 2a_\nu} \left|\frac{a_\mu+a_\nu+3}{a_\mu-a_\nu}\right|.$$

If $\rho(\nu)$ is the number of a_μ for which $a_\nu < a_\mu < 2a_\nu$, then

$$\left.\begin{aligned} \prod_{\mu<\nu} &< \frac{[2(a_\nu+2)]^{\nu-1}}{(\nu-1)!} < \left\{\frac{2(a_\nu+2)e}{\nu-1}\right\}^{\nu-1} \\ \prod_{a_\nu<a_\mu<2a_\nu} &< \frac{(3(a_\nu+2))^{\rho(\nu)}}{\rho(\nu)!} < \left\{\frac{3(a_\nu+2)e}{\rho(\nu)}\right\}^{\rho(\nu)} \end{aligned}\right\}. \tag{2.6}$$

Also,

$$\log(1+x) \leqslant x \quad (x \geqslant 0) \quad \text{and} \quad -\log(1-x) \leqslant 2x \quad (0 \leqslant x \leqslant \tfrac{1}{2}).$$

Hence

$$\prod_{a_\mu\geqslant 2a_\nu} = \prod \left\{\left(1+\frac{a_\nu+3}{a_\mu}\right)\left(1-\frac{a_\nu}{a_\mu}\right)^{-1}\right\} = \exp\left\{O\left(a_\nu \sum_{a_\mu\geqslant 2a_\nu} a_\mu^{-1}\right)\right\}. \tag{2.7}$$

As $\nu\to\infty$,

$$\nu/a_\nu \to 0, \qquad \rho(\nu)/a_\nu \to 0 \quad \text{and} \quad \sum_{a_\mu\geqslant 2a_\nu} a_\mu^{-1} \to 0.$$

Since $\lim\limits_{y\to\infty} y^{1/y} = 1$, (2.3) follows from (2.6) and (2.7).

The proof of the lemma is now very simple. $\sum\limits_1^\infty C_\nu t^{a_\nu}$ converges for every positive t less than 1. By (2.4) and (2.5), $P_n(t)$ tends to $\sum\limits_1^\infty \Lambda_\nu t^{a_\nu}$ uniformly in $0 \leqslant t \leqslant 1-\delta$ for every $\delta > 0$ (majorized convergence). Therefore

$$\int_0^{1-\delta} |\alpha(t) - \textstyle\sum \Lambda_\nu t^{a_\nu}|\,dt = \lim_{n\to\infty} \int_0^{1-\delta} |\alpha(t)-P_n(t)|\,dt = 0.$$

It follows that $$\alpha(t) = \textstyle\sum \Lambda_\nu t^{a_\nu} \quad \text{(p.p.)}$$

and the lemma is proved.

3. In this section I show that Theorem 2 is a consequence of Lemma 4 stated below.

I use the same notation as in the statement of Theorem 2. Considering real and imaginary parts separately we may suppose that the moments of T_1 and T_2 are real. If $T_2 \supseteq T_1$, then $\mu_n^{(1)} = 0$ implies $\mu_n^{(2)} = 0$, by Lemma 1 of (6). We can therefore find a set of constants γ_n satisfying

$$\mu_n^{(2)} = \gamma_n \mu_n^{(1)} \quad (n = 0, 1, 2, \ldots). \tag{3.1}$$

When $n \neq a_\nu$ $(\nu = 1, 2, \ldots)$, γ_n is uniquely determined; when $n = a_\nu$, γ_n may have any value.

If $S \sim \sigma_n$ is any Hausdorff method, $T_2 \supseteq T_1$ implies $T_2 S \supseteq T_1 S$. For the S-transform of any sequence is summed by T_2, if it is summed by T_1. In particular, let $T \sim \mu_n$ be any regular Hausdorff method with $\mu_n = 0$ for $n = a_1, a_2, \ldots$.† If the σ_n are chosen so that

$$\mu_n = \mu_n^{(1)} \sigma_n \quad (n = 0, 1, 2, \ldots),$$

then $T_1 S \sim \mu_n^{(1)} \sigma_n = \mu_n$ is regular and therefore

$$T_2 S \sim \mu_n^{(2)} \sigma_n = \gamma_n \mu_n^{(1)} \sigma_n = \gamma_n \mu_n$$

is also regular. In other words, the sequence $\{\gamma_n\}$ has the property

[A] *Whenever $T \sim \mu_n$ is regular and $\mu_n = 0$ $(n = a_1, a_2, \ldots)$, then $T' \sim \gamma_n \mu_n$ is also regular.*

Theorem 2 is a consequence of

LEMMA 4. *If the sequence $\{\gamma_n\}$ has the property* [A], *then*

$$\gamma_n = \int t^n \, d\phi(t) \quad (n \neq a_\nu).$$

For, by (3.1), $T_2 = \Theta T_1$, where $\Theta \sim \int t^n \, d\phi$ is regular.

4. Since the proof of Lemma 4 is fairly long, I first give an outline of it, postponing all details to later sections.

In order to prove Lemma 4 we study the transformation $\mathfrak{M} : \mu_n \to \gamma_n \mu_n$. By hypothesis, $\mathfrak{M}$ associates with every function ψ of V^* (moment constants μ_n) another function $\chi = \chi_\psi$ of V^* (moment constants $\gamma_n \mu_n$). If χ is normalized by

$$\chi(t) = \tfrac{1}{2}[\chi(t-0) + \chi(t+0)], \qquad \chi(1) = 0,$$

then χ_ψ is uniquely determined by the uniqueness theorem for the moment problem in a finite interval. The transformation $\psi \to \chi$ is obviously distributive and homogeneous, i.e.

$$\chi_{\psi+\psi'} = \chi_\psi + \chi_{\psi'}, \qquad \chi_{c\psi} = c\chi_\psi.$$

† Such methods exist, apart from the trivial one $\mu_n = 0$ $(n \geq 1)$. See (2).

I shall prove that it is also continuous. This is equivalent to proving that there is a constant M such that

$$\int |d\chi| \leqslant M \int |d\psi|. \tag{4.1}$$

It follows that, for any fixed x $(0 \leqslant x \leqslant 1)$, $\chi(x) = \chi(x;\psi)$ is a linear functional of ψ in V^* with norm at most M. In particular it is a linear functional $\chi(x;g)$ in the linear sub-space G^* of V^*. But G^* is also a sub-space of the space G. By a known theorem the functional $\chi(x;g)$ can be extended to a functional $A(x;g)$ of the same norm defined in the whole space G.† By Lemma 2 it can be represented in the form

$$A(x;g) = \int \alpha(x,t)g'(t)\,dt, \tag{4.2}$$

where $\alpha(x,t)$ is a measurable function of t for every x in $(0,1)$ and

$$|\alpha(x,t)| \leqslant M \quad \text{(p.p. in } t \text{ for every } x). \tag{4.3}$$

For $g(t)$ of G^*
$$A(x;g) = \chi(x;g). \tag{4.4}$$

The functional $A(x;g)$ is not uniquely determined. We shall find that it is possible to choose it in such a way that $\alpha(x,t)$ is a function of x/t only, i.e.

$$\alpha(x,t) = \beta(x/t), \tag{4.5}$$

where
$$\beta(w) = 0 \quad (w \geqslant 1). \tag{4.6}$$

With this special choice of $\alpha(x,t)$,

$$A(x) = A(x;g) = \int_x^1 \beta(x/t)g'(t)\,dt \quad (g \in G). \tag{4.7}$$

I shall show that
$$\int |dA| \leqslant M \int |g'|\,dt. \tag{4.8}$$

Multiplying (4.7) by $-kx^{k-1}$ $(k \geqslant 1)$ and integrating with respect to x from 0 to 1 gives

$$\begin{aligned}
\int x^k\,dA(x) &= -k \int x^{k-1}A(x)\,dx \\
&= -k \int x^{k-1}\,dx \int_x^1 \beta(x/t)g'(t)\,dt \\
&= -k \int g'(t)\,dt \int_0^t x^{k-1}\beta(x/t)\,dx \quad \text{(by Fubini's theorem)} \\
&= -k \int g'(t)t^k\,dt \int w^{k-1}\beta(w)\,dw \\
&= \tilde{\gamma}_k\mu_k \quad (k \geqslant 1),
\end{aligned} \tag{4.9}$$

† (1), 55.

where μ_k is the kth moment constant of g. If g is in G^*, $\int x^k\,dA(x)$ is the kth moment constant of $A(x) = \chi(x)$, so that

$$\int x^k\,dA(x) = \gamma_k\mu_k.$$

Hence $$\tilde{\gamma}_k = \gamma_k \quad (k \neq a_\nu;\ k \geqslant 1).$$

Lemma 4 will be proved if we can show that

$$\tilde{\gamma}_k = \int t^k\,da(t) \quad (k \geqslant 1) \tag{4.10}$$

(changing the value of $a(0)$, if necessary, we can make $\int da(t) = \gamma_0$). Let

$$g_n(t) = \begin{cases} -1 & (0 \leqslant t \leqslant 1-1/n), \\ n(1-t) & (1-1/n \leqslant t \leqslant 1), \end{cases}$$

and write $A_n(x)$ for $A(x; g_n)$. By (4.7) and (4.8), $A_n(1) = 0$ and $\int |dA_n| \leqslant M \int |g_n'|\,dt = M$. By a well-known theorem due to Helly† there is a sub-sequence $\{A_{n'}\}$ of the A_n tending to a function $a(x)$ of total variation at most M. Obviously $a(1) = 0$. By bounded convergence

$$\int x^k\,dA_{n'}(x) = -k\int x^{k-1}A_{n'}(x)\,dx \to -k\int x^{k-1}a(x)\,dx = \int x^k\,da(x). \tag{4.11}$$

But, by (4.9), $$\int x^k\,dA_{n'}(x) = \tilde{\gamma}_k \int t^k g_n'(t)\,dt, \tag{4.12}$$

and it is easily seen that

$$\int t^k g_n'(t)\,dt \to 1 \quad \text{as } n \to \infty.$$

Hence (4.10) follows from (4.11) and (4.12).

It remains to prove (4.1), (4.5), (4.6), (4.8).

5. Proof of (4.1). We write V_1^* for the set of ψ belonging to V^* with $\int |d\psi| \leqslant 1$. Since $\mathfrak{M}$ is homogeneous, it is enough to prove that

$$\int |d\chi| \leqslant M$$

for ψ of V_1^*.

Let $\mathfrak{B}$ be the set of polynomials $P(t) = \sum \lambda_k t^k$ with $|P(t)| \leqslant 1$ $(0 \leqslant t \leqslant 1)$. The hypothesis of Lemma 4 can be stated as follows. If $\psi \in V_1^*$, $\mu_n = \int t^n\,d\psi$, then

$$\operatorname*{up.\,bd.}_{P \in \mathfrak{B}} |F(P, \psi)| = \operatorname*{up.\,bd.}_{P \in \mathfrak{B}} |\textstyle\sum \lambda_k \gamma_k \mu_k| = M(\psi) < \infty. \tag{5.1}$$

† See (9), 29.

For, in virtue of Lemma 1 (ii), (5.1) is a necessary and sufficient condition for the existence of a function $\chi(t)$ with

$$\gamma_n \mu_n = \int t^n \, d\chi, \qquad \int |d\chi| \leqslant M(\psi).$$

We must prove $M(\psi) \leqslant M$.

Suppose this were not the case. Then there is a P_n of $\mathfrak{B}$ and a ψ_n of V_1^* such that for given M_n

$$|F(P_n, \psi_n)| \geqslant M_n. \tag{5.2}$$

We may also suppose that

$$|F(P_n, \psi_n)| > \tfrac{1}{2} \operatorname*{up.\,bd.}_{\psi \in V_1^*} |F(P_n, \psi)|. \tag{5.3}$$

For the right-hand side is certainly finite and, if (5.3) is not true to start with, we can replace ψ_n by another ψ satisfying (5.3).

Let $\alpha_\nu > 0$,

$$\sum_1^\infty \alpha_\nu = 1, \qquad \sum_{n+1}^\infty \alpha_\nu < \tfrac{1}{6}\alpha_n.$$

We choose in succession $M_1, \psi_1, P_1, \ldots, M_n, \psi_n, P_n, \ldots$, so that (5.2) and (5.3) are satisfied for every n and at the same time

$$\alpha_n M_n \to \infty, \qquad M_n > 3\alpha_n^{-1} \max_{\mu<n}\{M(\psi_\mu)\},$$

where $M(\psi_\mu)$ is defined by (5.1) with $\psi = \psi_\mu$. Let $\eta = \sum_1^\infty \alpha_\nu \psi_\nu$. Obviously $\eta \in V_1^*$.

$$\begin{aligned}
|F(P_n, \eta)| &= \Big|F(P_n, \alpha_n\psi_n) + F\Big(P_n, \sum_1^{n-1} \alpha_\nu \psi_\nu\Big) + F\Big(P_n, \sum_{n+1}^\infty \alpha_\nu \psi_\nu\Big)\Big| \\
&\geqslant |F(P_n, \alpha_n \psi_n)| - \sum_{\nu=1}^{n-1} |F(P_n, \alpha_\nu \psi_\nu)| - \sum_{\nu=n+1}^{\infty} |F(P_n, \alpha_\nu \psi_\nu)| \\
&\geqslant \alpha_n |F(P_n, \psi_n)| - \max_{\mu<n}\{M(\psi_\mu)\} \sum_1^{n-1} \alpha_\nu - 2|F(P_n, \psi_n)| \sum_{n+1}^\infty \alpha_\nu \\
&\geqslant \tfrac{2}{3}\alpha_n |F(P_n, \psi_n)| - \tfrac{1}{3}\alpha_n M_n \geqslant \tfrac{1}{3}\alpha_n M_n \to \infty
\end{aligned}$$

as $n \to \infty$. This contradicts $|F(P_n, \eta)| \leqslant M(\eta)$, and (4.1) is proved.

6. Proof of (4.5). If the transformation $\mathfrak{M}$ transforms the function $g(t)$ of G^* (moment constants μ_n) into the function $\chi(x)$ of V^* (moment constants $\gamma_n \mu_n$), then it transforms

$$g_u(t) = \begin{cases} g(t/u) & (t \leqslant u), \\ 0 & (u \leqslant t < 1) \end{cases} \quad \text{into} \quad \chi_u(x) = \begin{cases} \chi(x/u) & (x \leqslant u), \\ 0 & (u < x). \end{cases} \tag{6.1}$$

For g_u has the moment constants $\mu_n u^n$, and χ_u the moment constants $\gamma_n \mu_n u^n$. By (4.2) and (4.4),

$$\begin{aligned} \chi_u(ux) = \chi(x) &= \int_0^u \alpha(ux, t) g'_u(t)\, dt \\ &= \int \alpha(ux, ut) g'(t)\, dt. \end{aligned} \tag{6.2}$$

Subtracting this from (4.2) we find that

$$\int \{\alpha(x,t) - \alpha(ux, ut)\} g'(t)\, dt = 0 \quad (g \in G^*). \tag{6.3}$$

By Lemma 3, for fixed x and u,

$$\alpha(x,t) - \alpha(ux, ut) = \sum_1^\infty \lambda_\nu(u,x) t^{a_\nu} \quad \text{(p.p. in } t\text{)} \tag{6.4}$$

and $$|\lambda_\nu(u,x)| \leqslant 2MC_\nu.$$

Replacing x by ux, t by ut, and u by v in (6.4) gives

$$\alpha(ux, ut) - \alpha(uvx, uvt) = \sum_1^\infty \lambda_\nu(v, ux) u^{a_\nu} t^{a_\nu} \quad \text{(p.p.)} \tag{6.5}$$

for fixed x, u, v. Adding (6.4) and (6.5), we get

$$\begin{aligned} \alpha(x,t) - a(uvx, uvt) &= \sum \lambda_\nu(uv, x) t^{a_\nu} \\ &= \sum \{\lambda_\nu(u,x) + \lambda_\nu(v, ux) u^{a_\nu}\} t^{a_\nu} \quad \text{(p.p.)}. \end{aligned}$$

This is possible only if, for every ν,

$$\lambda_\nu(uv, x) - \lambda_\nu(u, x) = \lambda_\nu(v, ux) u^{a_\nu}. \tag{6.6}$$

Hence $|\lambda_\nu(uv,x) - \lambda_\nu(u,x)| < 2MC_\nu u^{a_\nu} < \epsilon$ for $u < u(\nu, \epsilon)$ and all $v < 1$. By Cauchy's test

$$\lim_{u \to 0} \lambda_\nu(u, x) = \lambda_\nu(x)$$

exists for every x. Also

$$\lim_{u \to 0} \{\alpha(x,t) - \alpha(ux, ut)\} = \lim \sum \lambda_\nu(u,x) t^{a_\nu} = \sum \lambda_\nu(x) t^{a_\nu} \quad \text{(p.p.)},$$

by majorized convergence. Letting v tend to 0 in (6.6) we have

$$\lambda_\nu(u, x) = \lambda_\nu(x) - \lambda_\nu(ux) u^{a_\nu}.$$

Substituting this value in (6.4) gives, for almost all t,

$$\alpha(x,t) - \sum \lambda_\nu(x) t^{a_\nu} = \alpha(ux, ut) - \sum \lambda_\nu(ux)(ut)^{a_\nu}. \tag{6.7}$$

The value of $\alpha(x,t)$ may be altered in a null set of t, for a fixed x, since this does not affect the functional (4.2). I show that $\alpha(x,t)$ can be changed in a null set of t for every x in such a way that

$$\alpha(x,t) - \sum \lambda_\nu(x) t^{a_\nu} = \beta(x/t) \tag{6.8}$$

is a function of x/t only. First, let t belong to the interval $I_0 : t < x$. In (6.7) replace x by 1, t by t/x, u by x. Then

$$\alpha(x,t) - \sum \lambda_\nu(x)t^{a_\nu} = \alpha(1,t/x) - \sum \lambda_\nu(1)(t/x)^{a_\nu} \quad (t \in I_0), \tag{6.9}$$

provided that $\alpha(x,t)$ is suitably changed in a null set $E_0 \subset I_0$. Similarly in $I_n : 2^{n-1}x \leqslant t < 2^n x$

$$\alpha(x,t) - \sum \lambda_\nu(x)t^{a_\nu} = \alpha(2^{-n}, 2^{-n}t/x) - \sum \lambda_\nu(2^{-n})(2^{-n}t/x)^{a_\nu} \quad (t \in I_n), \tag{6.10}$$

provided that $\alpha(x,t)$ is changed in a null set $E_n \subset I_n$. Since $E = \sum E_n$ is a null set, (6.9) and (6.10) prove (6.8).

For every value of x $(0 \leqslant x \leqslant 1)$ $\beta(x/t)$ is a measurable function of t $(0 < t \leqslant 1)$. It follows at once that $\beta(w)$ is measurable in $w \geqslant 0$.

Replacing x by ux, t by ut in (6.8) and letting u tend to 0 through an enumerable sequence of values, we find from (6.8) and (4.3) that

$$|\beta(x/t)| \leqslant M \quad \text{(p.p in } t\text{)}. \tag{6.11}$$

Also, for g of G^*,

$$\begin{aligned} \int \alpha(x,t)g'(t)\,dt &= \int \alpha(ux,ut)g'(t)\,dt, \quad \text{(by (6.3))}, \\ &= \int \{\beta(x/t) + \sum \lambda_\nu(ux)u^{a_\nu}t^{a_\nu}\}g'(t)\,dt \\ &= \int \beta(x/t)g'(t)\,dt + \sum \lambda_\nu(ux)u^{a_\nu} \int t^{a_\nu}g'(t)\,dt \\ &= \int \beta(x/t)g'(t)\,dt, \end{aligned} \tag{6.12}$$

by the definition of G^*. The term-by-term integration of the series is legitimate by uniform convergence. These results (6.12) and (6.11) show that we may replace $\alpha(x,t)$ in (4.2) by $\beta(x/t)$.

7. Proof of (4.6). If g_u is defined by (6.1), then, when $x > u$,

$$\begin{aligned} \chi_u(x) = 0 &= u^{-1} \int_0^u \beta(x/t)g'(t/u)\,dt \\ &= \int \beta(x/ut)g'(t)\,dt, \end{aligned}$$

i.e.

$$\int \beta(x/t)g'(t)\,dt = 0 \quad (x > 1,\ g \in G^*).$$

By Lemma 3,

$$\begin{aligned} \beta(x/t) &= \sum \Lambda_\nu(x)t^{a_\nu} \qquad \text{(p.p. in } t;\ x > 1) \\ &= \sum \Lambda_\nu(vx)v^{a_\nu}t^{a_\nu} \quad \text{(p.p. in } t;\ x^{-1} < v < 1) \end{aligned} \tag{7.1}$$

and

$$|\Lambda_\nu(y)| < MC_\nu \quad (y > 1).$$

Hence $$\Lambda_\nu(vx)(vx)^{a_\nu} = \Lambda_\nu(x)x^{a_\nu}.$$

Let $y = vx$ tend to 1. Then

$$\lim_{v\to 1}\Lambda_\nu(y) = \Lambda_\nu = \Lambda_\nu(x)x^{a_\nu}, \qquad \Lambda_\nu(x) = \Lambda_\nu x^{-a_\nu}.$$

Using this in (7.1) we have

$$\beta(x/t) = \sum \Lambda_\nu(t/x)^{a_\nu} \quad \text{(p.p. in } t;\ x>1),$$

i.e. $$\beta(w) = \sum \Lambda_\nu w^{-a_\nu} \quad \text{(p.p.; } w>1). \tag{7.2}$$

Now, if $g \in G^*$, then, by Fubini's theorem,

$$\begin{aligned} -a_\nu \int g'(t)\,dt \int \beta(x/t)x^{a_\nu-1}\,dx &= -a_\nu \int x^{a_\nu-1}\,dx \int \beta(x/t)g'(t)\,dt \\ &= -a_\nu \int x^{a_\nu-1}\chi(x)\,dx \\ &= \int x^{a_\nu}\,d\chi(x) = \gamma_{a_\nu}\mu_{a_\nu} = 0. \end{aligned}$$

By Lemma 3,

$$\int \beta(x/t)x^{a_\nu-1}\,dx = \sum_{\mu>1}^{\infty} k_\mu(\nu)t^{a_\mu} \quad \text{(p.p. in } t), \tag{7.3}$$

for any ν. But

$$\begin{aligned} \int \beta(x/t)x^{a_\nu-1}\,dx &= t^{a_\nu}\int_0^{1/t} \beta(w)w^{a_\nu-1}\,dw \\ &= t^{a_\nu}\left\{\int_0^2 + \int_2^{1/t}\right\} \quad (t<\tfrac{1}{2}) \\ &= t^{a_\nu}\{\text{const.} - \Lambda_\nu \log t\} + \sum_{\mu\neq\nu}\frac{\Lambda_\mu}{a_\nu - a_\mu}t^{a_\mu}, \end{aligned} \tag{7.4}$$

using (7.2)

By a simple uniqueness theorem for expansions of the form

$$\sum_1^\infty t^{a_\mu}\sum_0^r c_{\mu\rho}(\log t)^\rho,$$

(7.3) and (7.4) imply $$\Lambda_\nu = 0.$$

This is true for every value of ν and so, by (7.2),

$$\beta(w) = 0 \quad \text{(p.p.; } w>1),$$

which is (4.6).

8. Proof of (4.8). Let $g(t)$ belong to G^* and let $\chi(x)$ be the transform of g by $\mathfrak{M}$. For any division

$$\mathfrak{H}: 0 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$$

and any choice of signs $\epsilon_r = \pm 1$,

$$\Big|\sum_1^n \epsilon_r\{\chi(x_r)-\chi(x_{r-1})\}\Big| \leqslant \sum_1^n |\chi(x_r)-\chi(x_{r-1})| < M\,\|g\|,$$

by (4.1). For fixed $\mathfrak{H}$ the first sum is a linear functional in G^* and, by the argument leading to (4.2),

$$\sum_1^n \epsilon_r\{\chi(x_r)-\chi(x_{r-1})\} = \int \eta(\mathfrak{H},t)g'(t)\,dt,$$
$$|\eta(\mathfrak{H},t)| \leqslant M \quad \text{(p.p.)}.$$

Also, by (4.2) and (4.5),

$$\sum_1^n \epsilon_r\{\chi(x_r)-\chi(x_{r-1})\} = \int \sum \epsilon_r\{\beta(x_r/t)-\beta(x_{r-1}/t)\}g'(t)\,dt.$$

The expression

$$C(g) = \int \{\sum \epsilon_r(\beta(x_r/t)-\beta(x_{r-1}/t))-\eta(\mathfrak{H},t)\}g'(t)\,dt$$

is a linear function in G (Lemma 2) with

$$|C(g)| \leqslant (2n+1)M\,\|g\| \quad \text{and} \quad C(g) = 0 \quad (g \in G^*).$$

Hence, by Lemma 3,

$$\sum \epsilon_r\{\beta(x_r/t)-\beta(x_{r-1}/t)\} = \eta(\mathfrak{H},t)+\sum \lambda_\nu(\mathfrak{H})t^{a_\nu} \quad \text{(p.p. in } t\text{)},$$
$$|\lambda_\nu(\mathfrak{H},t)| < (2n+1)MC_\nu.$$

Replacing $\mathfrak{H}$ by $u\mathfrak{H}$, i.e. x_r by ux_r, we obtain

$$\sum \epsilon_r\{\beta(x_r/t)-\beta(x_{r-1}/t)\} = \eta(u\mathfrak{H},ut)+\sum \lambda_\nu(u\mathfrak{H})(ut)^{a_\nu} \quad \text{(p.p.)},$$

where $\quad |\eta| \leqslant M \quad \text{(p.p. in } t\text{)}, \qquad |\lambda_\nu| < (2n+1)MC_\nu.$

Letting u tend to 0 through an enumerable sequence we find that

$$|\sum \epsilon_r\{\beta(x_r/t)-\beta(x_{r-1}/t)\}| \leqslant M \quad \text{(p.p. in } t\text{)}.$$

Since this is true for all 2^n choices of the ϵ_r,

$$\sum |\beta(x_r/t)-\beta(x_{r-1}/t)| \leqslant M \quad \text{(p.p. in } t\text{)}. \tag{8.1}$$

By (4.2), (4.5), (8.1)

$$\begin{aligned}\int |dA| &= \text{up. bd.} \sum |A(x_r)-A(x_{r-1})| \\ &\leqslant \text{up. bd.} \int \sum |\beta(x_r/t)-\beta(x_{r-1}/t)|\,|g'(t)|\,dt \\ &\leqslant M\,\|g\|,\end{aligned}$$

and (4.8) is proved.

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SOME SOLUTIONS OF THE HODOGRAPH EQUATION GOVERNING THE TWO-DIMENSIONAL FLOW OF A COMPRESSIBLE FLUID

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In general, the region of steady two-dimensional flow of a compressible fluid consists of both subsonic and supersonic regions, which are such that in the former the speed of the fluid is at each point less than the local speed of sound, while in the latter the opposite is the case. Thus the curve which separates these regions, the so-called *sonic curve*, is the locus of points in the plane at which the local speed of sound is attained by the fluid. When the spatial coordinates are the independent variables, and the pressure-density relationship is adiabatic, the differential equations for the stream-function and velocity-potential are both of the second order and non-linear. Also they are elliptic or hyperbolic in type according as the fluid motion is subsonic or supersonic.

In the two-dimensional hodograph method, introduced in a memoir by Tchapliguine* and later applied to particular types of problems on subsonic motion, the independent variables are either the two components of velocity, or the speed and an angle defining direction of motion at the point concerned. The resulting second-order differential equations for the stream-function and velocity-potential are both linear. Their general solutions are not known, but some interesting particular solutions have been discussed† with the object in view of investigating the properties of the associated fluid motion.

In the present application of the hodograph method the two differential equations mentioned above are briefly deduced in § 1, as is also a normal solution of importance in particular problems.‡ In connexion with the normal solution original equations expressing the correspondence between the plane of cartesian coordinates and the hodograph plane are obtained in § 2. Finally, in § 3, some further particular solutions for the stream-function are found, two of which are described in some detail. In one case the associated motion is

* *Scientific Ann. Univ. Moscow*, 21 (1904), 1–121.

† Cf. e.g., W. Tollmien, *Zeits. f. angew. Math. u. Mech.* 17 (1937), 117–36.

‡ Cf. e.g., B. Demtchenko, *Publ. Math. Univ. Belgrade*, 2 (1933).

entirely subsonic, in the other the field of flow may include also a limited supersonic region.

The notation employed is as follows.

The fluid velocity at (x, y) is v and is inclined to the x-axis at an angle θ; the stream-function and velocity-potential are ψ, ϕ respectively; the pressure is p, the density σ, where p/σ^γ is constant; $a = (\gamma p/\sigma)^{\frac{1}{2}}$ is the local speed of sound; a_0, σ_0 denote the speed of sound and the density where the local speed of sound is attained by the fluid, i.e. on the sonic curve. Also

$$\rho = \sigma/\sigma_0, \quad \lambda = q^2 = (v/a_0)^2, \quad \mu = (\gamma-1)\lambda/(\gamma+1), \quad \nu = 1/\rho\lambda^{\frac{1}{2}}.$$

1. Bernoulli's equation

$$\tfrac{1}{2}v^2 + \int \frac{dp}{\sigma} = \text{constant}$$

can be written in the form

$$2\rho^{\gamma-1} = 2a^2/a_0^2 = (\gamma+1)-(\gamma-1)\lambda. \tag{1.1}$$

The equation of continuity and the condition for irrotational motion are both satisfied if the velocity components are expressed in the form

$$\rho q\cos\theta = \frac{\partial\psi}{\partial y}, \qquad \rho q\sin\theta = -\frac{\partial\psi}{\partial x},$$

$$q\cos\theta = \frac{\partial\phi}{\partial x}, \qquad q\sin\theta = \frac{\partial\phi}{\partial y}.$$

Thus

$$d\phi = q\cos\theta\,dx + q\sin\theta\,dy,$$

$$d\psi = -\rho q\sin\theta\,dx + \rho q\cos\theta\,dy,$$

and therefore, if we consider ϕ and ψ to be functions of λ and θ,

$$dx = \lambda^{-\frac{1}{2}}\left\{\left(\phi_\lambda\cos\theta - \frac{1}{\rho}\psi_\lambda\sin\theta\right)d\lambda + \left(\phi_\theta\cos\theta - \frac{1}{\rho}\psi_\theta\sin\theta\right)d\theta\right\}, \tag{1.2}$$

$$dy = \lambda^{-\frac{1}{2}}\left\{\left(\phi_\lambda\sin\theta + \frac{1}{\rho}\psi_\lambda\cos\theta\right)d\lambda + \left(\phi_\theta\sin\theta + \frac{1}{\rho}\psi_\theta\cos\theta\right)d\theta\right\}, \tag{1.3}$$

where subscripts denote partial derivatives.

Since the right-hand sides of (1.2) and (1.3) must be perfect differentials, we have, on applying the usual conditions and simplifying,

$$\phi_\lambda = -\frac{(\gamma+1)(1-\lambda)}{4\lambda\rho^\gamma}\psi_\theta \tag{1.4}$$

and

$$\phi_\theta = \frac{2\lambda}{\rho}\psi_\lambda. \tag{1.5}$$

[The identities

$$\rho' = d\rho/d\lambda = -\tfrac{1}{2}\rho^{2-\gamma} \quad \text{and} \quad \nu' = d\nu/d\lambda = -(\gamma+1)(1-\lambda)/4\rho^{\gamma}\lambda^{\frac{1}{2}}$$

are useful in carrying out the above simplifications, and in subsequent calculations.]

Thus the differential equation for the stream-function is

$$\frac{\partial}{\partial\lambda}\left(\frac{\lambda}{\rho}\frac{\partial\psi}{\partial\lambda}\right)+\frac{(\gamma+1)(1-\lambda)}{8\lambda\rho^{\gamma}}\frac{\partial^2\psi}{\partial\theta^2}=0, \tag{1.6}$$

and that for the velocity-potential

$$\frac{\partial}{\partial\lambda}\left(\frac{\lambda\rho^{\gamma}}{1-\lambda}\frac{\partial\phi}{\partial\lambda}\right)+\frac{(\gamma+1)\rho}{8\lambda}\frac{\partial^2\phi}{\partial\theta^2}=0.$$

Obviously (1.6) has a solution of the form

$$\psi = A+B\theta+\sum\psi_n, \tag{1.7}$$

where

$$\psi_n = f_n(\lambda)\sin(2n\theta+\alpha_n),$$

and f_n is such that

$$\frac{d}{d\lambda}\left(\frac{\lambda}{\rho}f_n'\right)=\frac{n^2(\gamma+1)(1-\lambda)}{2\lambda\rho^{\gamma}}f_n. \tag{1.8}$$

Substituting $\lambda^n g_n(\lambda)$ for $f_n(\lambda)$ and μ for $(\gamma-1)\lambda/(\gamma+1)$, we have

$$\mu(1-\mu)g_n''+\left\{(2n+1)-\left(2n+1-\frac{1}{\gamma-1}\right)\mu\right\}g_n'+\frac{n(2n+1)}{\gamma-1}g_n=0, \tag{1.9}$$

where dashes denote differentiation with respect to μ.

Thus, if g_n is finite at $\mu=0$, $\lambda=0$, it must be proportional to the hypergeometric function

$$F(a_n, b_n; c_n; \mu), \tag{1.10}$$

where

$$a_n+b_n = 2n-\frac{1}{\gamma-1}, \qquad a_n b_n = -\frac{n(2n+1)}{\gamma-1}, \qquad c_n = 2n+1. \tag{1.11}$$

In any case, however, if $g(\mu)$ is one solution of (1.9), the other independent solution is

$$g\int\frac{\rho\,d\mu}{\mu^{2n+1}g^2}. \tag{1.12}$$

2. When the stream-function is that given by (1.7), the coordinates can be found by integration of (1.2) and (1.3). Substituting for ϕ_λ and ϕ_θ from (1.4) and (1.5), and ν for $1/\rho\lambda^{\frac{1}{2}}$, we have

$$dx = (\nu'\psi_\theta\cos\theta-\nu\psi_\lambda\sin\theta)\,d\lambda+(2\lambda\nu\psi_\lambda\cos\theta-\nu\psi_\theta\sin\theta)\,d\theta, \tag{2.1}$$

$$dy = (\nu'\psi_\theta\sin\theta+\nu\psi_\lambda\cos\theta)\,d\lambda+(2\lambda\nu\psi_\lambda\sin\theta+\nu\psi_\theta\cos\theta)\,d\theta. \tag{2.2}$$

The term $\psi_0 = A + B\theta$ occurring in ψ contributes to x and y respectively

$$x_0 = B\nu\cos\theta + \text{constant} \quad \text{and} \quad y_0 = B\nu\sin\theta + \text{constant}. \tag{2.3}$$

Now let

$$\psi = \psi_n = f_n(\lambda)\sin(2n\theta + \alpha_n).$$

After some simplification, (2.1) gives

$$dx = \{(\nu' n f_n + \tfrac{1}{2}\nu f_n')\cos(\overline{2n+1}\theta + \alpha_n) + (\nu' n f_n - \tfrac{1}{2}\nu f_n')\cos(\overline{2n-1}\theta + \alpha_n)\}\, d\lambda +$$
$$+ \{(\lambda\nu f_n' - \nu n f_n)\sin(\overline{2n+1}\theta + \alpha_n) + (\lambda\nu f_n'' + \nu n f_n)\sin(\overline{2n-1}\theta + \alpha_n)\}\, d\theta.$$

Thus the corresponding contribution to x is given by

$$x_n = X(\lambda) - \left\{(\lambda\nu f_n' - \nu n f_n)\frac{\cos(\overline{2n+1}\theta + \alpha_n)}{2n+1} + \right.$$
$$\left. + (\lambda\nu f_n' + \nu n f_n)\frac{\cos(\overline{2n-1}\theta + \alpha_n)}{2n-1}\right\}.$$

Since, from (1.8),

$$\frac{d}{d\lambda}(\lambda\nu f_n') = -\tfrac{1}{2}\nu f_n' - 2n^2\nu' f_n,$$

it is clear, when $\partial x_n/\partial\lambda$ as obtained from the last expression is identified with the coefficient of $d\lambda$ in the previous line, that X is a constant. In a similar way $y_n(\lambda, \theta)$ is determined. We have finally, corresponding to the stream-function (1.7),

$$x = x_0 + \sum x_n, \qquad y = y_0 + \sum y_n, \tag{2.4}$$

where x_0, y_0 are given by (2.3) and

$$\left.\begin{aligned} x_n &= -\nu\left\{(\lambda f_n' - n f_n)\frac{\cos(\overline{2n+1}\theta + \alpha_n)}{2n+1} + (\lambda f_n' + n f_n)\frac{\cos(\overline{2n-1}\theta + \alpha_n)}{2n-1}\right\} \\ y_n &= -\nu\left\{(\lambda f_n' - n f_n)\frac{\sin(\overline{2n+1}\theta + \alpha_n)}{2n+1} - (\lambda f_n' + n f_n)\frac{\sin(\overline{2n-1}\theta + \alpha_n)}{2n-1}\right\} \end{aligned}\right\} \tag{2.5}$$

the arbitrary additive constants involved in x_n, y_n being absorbed in those occurring in the expressions for x_0, y_0.

The stream-function (1.7) and the equations (2.4) giving the correspondence between the plane of cartesian coordinates and the hodograph plane are sufficient for the formal solution of the type of problem referred to in the introduction. In such a case n takes positive integral values and $f_n(\lambda)$ is to be expressed in terms of a hypergeometric function as explained above.

When $n = \frac{1}{2}$, the expressions for x_n, y_n cannot be obtained directly from (2.5) but they follow easily by integration of (2.1) and (2.2). These expressions are quoted in one of the following particular solutions.

3. Apart from the trivial solution, $\psi = \psi_0 = A + B\theta$, of (1.6), some particular solutions can be derived from an investigation of the hypergeometric equation (1.9). It may be mentioned, however, that the motion corresponding to $\psi = \psi_0$ and (2.3) is that due to a 'source', since the equation of continuity, $\nu(x_0^2+y_0^2)^{\frac{1}{2}} = \text{constant}$, and the condition for straight concurrent streamlines are satisfied.

The function (1.10) can be identified in a special case with the function

$$F(a, b; b; \mu) \equiv (1-\mu)^{-a}.$$

It will be seen from (1.11) that b_n and c_n are equal only when $n = \gamma$ (unless $n = -\frac{1}{2}$, in which case $b_n = c_n = 0$). Thus a solution corresponds to

$$n = \gamma, \qquad a_n = -\gamma/(\gamma-1), \qquad b_n = c_n = 2\gamma+1,$$

and $g_n(\mu) = (1-\mu)^{\gamma/(\gamma-1)}$, which is proportional to ρ^γ. In this case therefore the solution may be written

$$\psi = (\lambda\rho)^\gamma \sin 2\gamma\theta. \tag{3.1}$$

Another solution for this value of n can, by (1.12), be written

$$\psi = \Big(\rho^\gamma \int \lambda^{-1-2\gamma}\rho^{1-2\gamma}\, d\lambda\Big)\sin 2\gamma\theta. \tag{3.2}$$

The coordinates can be obtained in each case from (2.5), with $n = \gamma$ and the appropriate expression for $f_n(\lambda)$. Here (3.1) corresponds to motion within a closed region containing the origin, while that represented by (3.2) is inside a region which excludes the origin. In each case the family of streamlines is symmetrical with respect to the x-axis, and it includes, for $-\pi < \theta \leqslant \pi$, the concurrent straight lines whose directions are given by $\theta = \pm\pi/2\gamma, \pm\pi/\gamma$.

The only case in which (1.9) has a solution of the form μ^m is when $n = \frac{1}{2}$; the solution is then μ^{-1}. The corresponding solution of (1.8) is $\lambda^{-\frac{1}{2}}$. The second independent solution, given by (1.12), is

$$\mu^{-1} \int \rho\, d\mu, \qquad \text{i.e.} \quad \mu^{-1}(1-\mu)^{\gamma/(\gamma-1)},$$

which is proportional to $\lambda^{-1}\rho^\gamma$; the appropriate solution of (1.8) is $\lambda^{-\frac{1}{2}}\rho^\gamma$.

Evaluating the coordinates by means of (1.2) and (1.3), and

choosing arbitrary constants suitably, we may write these two particular solutions

$$\left.\begin{aligned} &\text{(i)} \quad \psi = \lambda^{-\frac{1}{2}}\rho^{\gamma}\sin\theta \\ &\text{(ii)} \quad x = (\gamma+1)\log\lambda + \left(1+\frac{\gamma+1}{\lambda}\right)\cos 2\theta \\ &\text{(iii)} \quad y = -2\gamma\theta + \left(1+\frac{\gamma+1}{\lambda}\right)\sin 2\theta \end{aligned}\right\}, \tag{3.3}$$

$$\left.\begin{aligned} &\text{(i)} \quad \psi = 2\lambda^{-\frac{1}{2}}\sin\theta \\ &\text{(ii)} \quad x = I(\lambda) + (\rho\lambda)^{-1}\cos 2\theta \\ &\text{(iii)} \quad y = (\rho\lambda)^{-1}\sin 2\theta \end{aligned}\right\}, \tag{3.4}$$

where

$$I(\lambda) \equiv \tfrac{1}{2}\int_{\lambda}^{1} \xi^{-1}\rho^{-\gamma}\, d\xi.$$

Any singular points on the curves $\psi =$ constant must lie also on the envelope of the curves $\lambda =$ constant (i.e. on the curves of constant speed), for the coordinates of these singular points are such that

$$x_\lambda y_\theta - x_\theta y_\lambda = 0,$$

which equation, together with (ii) and (iii), represents the above envelope. By means of (2.1) and (2.2) this equation can be put in the form

$$8\lambda^2\rho^{\gamma-1}\psi_\lambda^2 + (\gamma+1)(1-\lambda)\psi_\theta^2 = 0, \tag{3.5}$$

which shows that such singular points are possible only in the supersonic region, in which $\lambda > 1$.

In the discussion of the fluid motions represented by (3.3) and (3.4) the assumed value of γ is that for dry atmospheric air, namely 1·405. With regard to the solution (3.3), the streamlines on which ψ has positive constant values, and also curves on which λ is constant (i.e. λ-curves), are shown in Fig. 1; the figure is symmetrical about the line $y = -\pi\gamma$, on which $\theta = \frac{1}{2}\pi$.

The curves $\lambda =$ constant, the broken curves in the figure, are portions of trochoids and are shown for $\lambda = 1$, 2, 3 and $(\gamma+1)/(\gamma-1)$; the latter curve, a cycloid with cusps at C, determines the limit up to which Bernoulli's equation can apply. The relevant portion of the envelope, accentuated in the figure, has zero slope at the points C, and has cusps just inside the loops of the trochoid $\lambda = 1$. Its equation is given by (ii) and (iii) of (3.3) and, in accordance with (3.5),

$$\cos 2\theta = \frac{\gamma\lambda^2+\lambda+\gamma+1}{\lambda(\lambda+2\gamma+1)}.$$

The extent to which the motion may be supersonic whilst still irrotational is determined by the envelope of the λ-curves. Our theoretical results show that each streamline for which $\psi < 0{\cdot}15$,

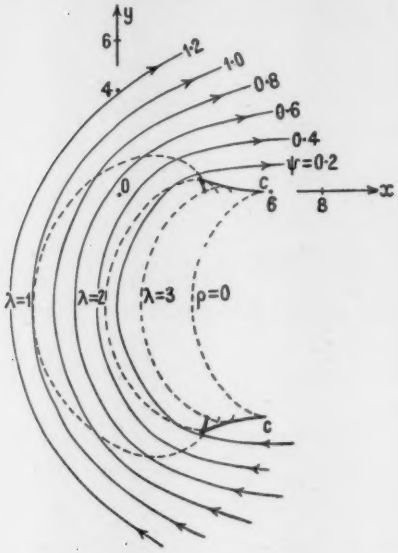

FIG. 1. λ-curves, their envelope, and the streamlines (Solution (3.3)).

approximately, meets the envelope twice and also intersects itself, such a streamline having a pair of cusps on the envelope (Fig. 2). This is clearly not possible physically. Either the assumption of irrotational flow ceases to be valid in the region lying within the

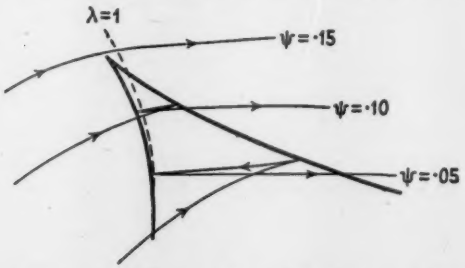

FIG. 2. Enlargement of Fig. 1 in the neighbourhood of the envelope.

cusp of the envelope, or the physical nature of the flow is such that discontinuities of pressure and speed occur at the envelope. Outside these forbidden regions, however, where (3.3) is applicable, there is a region in which the flow is supersonic and irrotational. The maximum value of λ in this region is 2·65, approximately, corresponding to $v/a = 2{\cdot}00$. This maximum specific speed occurs at the intersection of the streamline $\psi = 0{\cdot}15$ with $y = -\pi\gamma$, the axis of symmetry of Fig. 1.

Fig. 3 shows the streamlines $\psi = 2, 3, 4, 5$, and the curves for which $\lambda = 1{\cdot}0, 0{\cdot}6, 0{\cdot}4, 0{\cdot}2$, in the motion represented by (3.4). The λ-curves are the circles

$$\{x - I(\lambda)\}^2 + y^2 = (\rho\lambda)^{-2}.$$

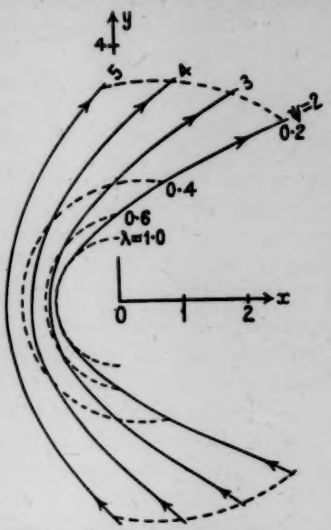

FIG. 3. λ-curves and streamlines (Solution (3.4)).

As λ increases from 0·2 to 2·0, $I(\lambda)$ diminishes as shown in the following table:

λ	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
I	0·583	0·365	0·221	0·104	0·000	−0·088	−0·194	−0·290	−0·390	−0·496

As λ increases from unity, the radius of the λ-circle diminishes from unity to a minimum when $\lambda = (\gamma+1)/\gamma$, and then increases steadily. The envelope of the λ-circles, symmetrical with respect to the x-axis, has cusps at the points for which $\lambda = 2$, and extends from them to $x = -\infty$, $y = \pm\infty$.

When $\lambda > 1$, the curves $\psi =$ constant therefore intersect the critical streamline $\psi = 2$ (see Fig. 3) and each other. Thus inside the region bounded by this streamline the flow represented by (3.4) is not physically possible; outside this region, where the latter equations are applicable, the flow is nowhere supersonic although the local speed of sound is just attained at the point $(-1, 0)$.

ON THE DEFINITION AND VALUE OF THE AREA OF A SURFACE

By A. S. BESICOVITCH (*Cambridge*)

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NEW methods in the theory of functions of a real variable have naturally brought the problem of the area of a surface to prominence. Schwarz had already discovered a fallacy in the definition of the area as the limit of the area of an inscribed polyhedron, showing that such a limit does not exist. However, as surfaces considered at that time were those with a continuously varying tangential plane, the fallacy in the definition never led to any fallacy in defining the actual value of the area, and the definition itself was amended in a satisfactory way.

With the new methods in the theory of functions, much more general classes of surfaces became subjects of study, and the old definition in its amended form became unsuitable. Then Lebesgue gave his definition of the area as the lower limit of areas of a sequence of polyhedra converging to the surface. At that moment the definition seemed to be very natural and satisfactory. The necessity of considering tangential planes was avoided and the definition was intuitively acceptable.

The definition was later modified by Fréchet. It was mainly meant for surfaces homeomorphic with a disk and did not aim at and was not suitable for giving some kind of measure to sets different from surfaces, for instance to Peano curves as was indicated by Geöcze.*

Then came the second phase in the development of the methods of measure. In 1914 Carathéodory defined p-dimensional measure in q-dimensional space, and he showed that the length of a rectifiable curve coincides with its one-dimensional measure.

A similar problem on the connexion of the area and the two-dimensional measure of a surface is of obvious importance and yet no step towards its solution has been made. In spite of extensive

* Rado has pointed out that in the case of a plane area whose boundary is a Jordan curve of positive plane measure the definition gives the value of the interior area.

and important work on the area of a surface that has been done since 1902 by Lebesgue, Geöcze, W. H. Young, Burkill, Tonelli, Fréchet, Rado, Morrey, McShane, Reichelderfer and others, the problem of the area has remained completely detached from that of measure. This gap by itself would be bad enough, but as a matter of fact the position is much worse.

In studying the relation between the values of Lebesgue-Fréchet area and the Carathéodory two-dimensional measure, I arrived first at cases where these two values are different. This alone would not be a serious matter, but then I constructed a surface homeomorphic with a disk, for which Lebesgue-Fréchet value of area is finite, and which considered as a set of points in the three-dimensional space has a positive (three-dimensional) Lebesgue measure. The same construction gives also an example of a solid homeomorphic to a sphere, whose surface has arbitrarily small area in the Lebesgue-Fréchet sense, and whose volume is arbitrarily large. In all these cases the surfaces can be given in a parametric form so that the L.-F. area is expressible by the well-known integral. This shows that the *Lebesgue-Fréchet definition cannot be retained and that all the results obtained from this definition have to be done anew.*

The position differs in each of the two classes of the problem. In the case of surfaces given parametrically the results have to be abandoned together with the definition, just because of their 'completeness'. The results for this class of surfaces obtained during the last ten years are that, whenever the surface has a finite Lebesgue-Fréchet area, it is expressible by the well-known integral. These results have to go, together with the definition.

In the case of surfaces given by the equation $z = f(x, y)$ I consider the set E_G of points $\{x, y, f(x, y)\}$ where G, a plane measurable set, is the projection of E_G on the plane $z = 0$. It is shown that, if the function $f(x, y)$ is absolutely continuous in the Tonelli sense, then, denoting by Λ^2 the Carathéodory two-dimensional measure, we have

$$\Lambda^2 E_G = \iint_G \sqrt{(1+f_x^2+f_y^2)}\, dxdy.$$

In particular when G is a rectangle we have the Tonelli case.

As a side-issue an example is given of a surface E ($\Lambda^2 E < \infty$) that has no kind of a generalized tangential plane at a set E_1 of its points such that $\Lambda^2 E_1 > 0$.

1. Let A be a cube of side unity. Divide it into 8 equal cubes and inside each of them construct a concentric cube with sides parallel to the sides of A and each one of volume $\frac{3}{4}.\frac{1}{8}$. Denote these cubes by A_i $(i = 1, 2,..., 8)$. These cubes are isolated from one another. Now divide each cube A_{i_1} into eight equal cubes and inside each of them construct a concentric cube with sides parallel to those of A and of volume $\frac{5}{8}.\frac{1}{8^2}$. Denote these cubes by $A_{i_1 i}$ $(i = 1, 2,..., 8)$. Similarly we construct cubes $A_{i_1 i_2 i}$ $(i = 1, 2,..., 8)$ of volume $\frac{9}{16}.\frac{1}{8^3}$, and so on. We have a decreasing sequence of sets

$$A, \qquad \sum_{i_1=1,...,8} A_{i_1}, \qquad \sum_{\substack{i_1=1,...,8 \\ i_2=1,...,8}} A_{i_1 i_2}, \ldots .$$

Let D be the limit of this sequence. The three-dimensional Lebesgue measure of D is obviously $\frac{1}{2}$. We shall denote as usual by $|E|$ the Lebesgue measure of the set E, so that it is one-dimensional measure in case of a linear set, two-dimensional measure in case of a plane set, and so on. We can write

$$|D| = \tfrac{1}{2}.$$

On a side of the cube A take a square B and inside B eight isolated squares C_i $(i = 1, 2,..., 8)$. Similarly on a side of the cube $A_{i_1 i_2 \ldots i_k}$ take a square $B_{i_1 i_2 \ldots i_k}$ and inside it eight isolated squares $C_{i_1 i_2 \ldots i_k i}$ $(i = 1, 2,..., 8)$. We write

$$B' = B - \sum_{i=1}^{8} C_i, \qquad B'_{i_1 i_2 \ldots i_k} = B_{i_1 i_2 \ldots i_k} - \sum_{i=1}^{8} C_{i_1 i_2 \ldots i_k i},$$

so that $B'_{i_1 i_2 \ldots i_k}$ is a square with eight square holes in it. Join every square C_i to the square B_i by a pipe c_i whose surface consists of plane quadrangles and triangles, so that all c_i are inside A and outside A_i $(i = 1, 2,..., 8)$ and do not meet. Similarly, join every square $C_{i_1 \ldots i_k i}$ to the square $B_{i_1 \ldots i_k i}$ by a pipe $c_{i_1 \ldots i_k i}$, so that all the eight pipes $c_{i_1 \ldots i_k i}$ $(i = 1, 2,..., 8)$ lie inside $A_{i_1 \ldots i_k}$ and outside $A_{i_1 \ldots i_k i}$ $(i = 1, 2,..., 8)$ and do not meet. Consider the surface

$$P' = B' + \sum_{i_1=1,...,8} c_{i_1} + \sum_{i_1=1,...,8} B'_{i_1} + \sum_{\substack{i_1=1,...,8 \\ i_2=1,...,8}} c_{i_1 i_2} + \sum_{\substack{i_1=1,...,8 \\ i_2=1,...,8}} B'_{i_1 i_2} + \ldots .$$

Obviously we can construct the squares and the pipes so that the area of P' may be as small as we please.

P' is not a closed set of points. Denote by P its closure so that $P = P' + P''$. No point outside any of the cubes $A_{i_1 \ldots i_k}$ belongs to P'', since the part of P' outside the cubes $\sum A_{i_1 \ldots i_k}$ is a polyhedral surface that has no limit points outside itself. On the other hand, each one of our cubes has points in common with P' and, therefore, every point of D is a limit-point of P'. Thus $P'' = D$. Therefore *The three-dimensional Lebesgue measure of P is $\frac{1}{2}$, i.e. $|P| = \frac{1}{2}$.*

The set P is a surface homeomorphic with a square. We shall define a topological representation of P on a square. Let Γ be a square of side unity. Construct on it eight isolated pairs of concentric squares $\overline{\Gamma}_i$, Γ_i $(i = 1, 2, \ldots, 8)$, Γ_i inside $\overline{\Gamma}_i$. Similarly on each square Γ_{i_1} construct eight isolated pairs of concentric squares $\overline{\Gamma}_{i_1 i}$, $\Gamma_{i_1 i}$ $(i = 1, 2, \ldots, 8)$, and so on. Write

$$\Gamma' = \Gamma - \sum_{i=1,\ldots,8} \overline{\Gamma}_i, \qquad \Gamma'_{i_1 \ldots i_k} = \Gamma_{i_1 \ldots i_k} - \sum_{i=1,\ldots,8} \overline{\Gamma}_{i_1 \ldots i_k i}$$

so that Γ', $\Gamma'_{i_1 \ldots i_k}$ are squares with eight square holes in each of them. We shall assume that the total area of $\sum \Gamma_{i_1 \ldots i_k}$ tends to 0 as $k \to \infty$. Write further

$$\Gamma = \Pi' + \Pi'',$$

where

$$\Pi'' = \lim_{k \to \infty} \sum \Gamma_{i_1 \ldots i_k}, \qquad |\Pi''| = 0.$$

We shall call a topological representation of a set U on a set V *quasi-linear* if V can be represented as the sum $\sum V_n$, so that, on each V_n, coordinates of points of U are linear functions of the coordinates of the corresponding points of V_n. Obviously there exist quasi-linear representations of B' on Γ'. Take one of them. It fixes a correspondence on outer and inner boundaries of B' and Γ'. We can now define a quasi-linear representation of every pipe c_i on the figure $\overline{\Gamma}_i - \Gamma_i$, so that on the outer boundary of $\overline{\Gamma}_i$ it coincides with the one that has been established before. We proceed similarly establishing a quasi-linear representation of B'_{i_1} on Γ'_{i_1}, of $c_{i_1 i_2}$ on $\overline{\Gamma}_{i_1 i_2} - \Gamma_{i_1 i_2}$, of $B'_{i_1 i_2}$ on $\Gamma'_{i_1 i_2}$ and so on, always taking care that the consecutive representations coincide on the boundaries. In this way P' is represented on Π'. Representation of P'' on Π'' is extended by the continuity principle.

Denoting by u, v coordinates of points of Γ, let the functions

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v)$$

express the above representation. These functions are finitely differentiable at almost all points of Γ, and the integral

$$\iint \sqrt{\left[\left\{\frac{\partial(x,y)}{\partial(u,v)}\right\}^2+\ldots\right]}\,dudv$$

extended over Γ represents the area of P'.

We shall now define a sequence P_n of polyhedral surfaces in the following way. Write

$$P'_n = B' + \sum c_{i_1} + \sum B'_{i_1} + \sum c_{i_1 i_2} + \sum B'_{i_1 i_2} + \ldots + \sum c_{i_1 \ldots i_n} + \sum B'_{i_1 \ldots i_n},$$

$$P_n = P'_n + \sum B_{i_1 \ldots i_n i_{n+1}}.$$

The additional sum in P_n seals the holes of P'_n, so that P_n is a continuous polyhedral surface, whose boundary is that of the square B and consequently that of the surface P. We shall represent P_n on Γ by representing P'_n in the same way as was done in the case of the surface P and then representing the squares $B_{i_1 \ldots i_n i_{n+1}}$ linearly on the squares $\Gamma_{i_1 \ldots i_{n+1}}$. Let the functions of the representation be

$$x = f_n(u,v), \qquad y = g_n(u,v), \qquad z = h_n(u,v).$$

Obviously the sequences of the functions $f_n(u,v)$, $g_n(u,v)$, $h_n(u,v)$ tend uniformly to $f(u,v)$, $g(u,v)$, $h(u,v)$. The area of P_n tends to that of P' and therefore, according to the Lebesgue-Fréchet definition, the area of P is less than or equal to that of P' and thus is finite. Thus the Lebesgue-Fréchet definition leads to a position where a *surface, whose three-dimensional Lebesgue measure is equal to* $\frac{1}{2}$, *has a finite area.*

Another consequence of the definition is this. Close the surface P from the top by a pyramid of base B and of small height. We shall get a closed surface. *The area of the surface in the Lebesgue-Fréchet sense may be as small as we please and yet the volume is greater than* $\frac{1}{2}$.

Remark. We shall now modify the construction of the surface P. We take the same cube as before and we construct 8 cubes A_{i_1} in the same way as before, except that we shall change their size, so as to have their total surface equal to that of A. Similarly in each A_{i_1} we construct 3^3 cubes $A_{i_1 i_2}$, then in each $A_{i_1 i_2}$ we construct 4^3 cubes $A_{i_1 i_2 i_3}$ and so on, always so that the total surface of all $A_{i_1 i_2 \ldots i_n}$ is equal to that of A. As before denote by D the limit of the set

$$\sum_{1 \leq i_k \leq (k+1)^3} A_{i_1 i_2 \ldots i_n}$$

as $n \to \infty$. In this case $\Lambda^2 D$ is positive and finite. In fact it lies between the limits $\frac{1}{2}\pi$ and $\frac{1}{4}\pi$. Now the surface $P = P' + P''$ is defined with the obvious modifications. As before $\Lambda^2 P'$, and consequently $\Lambda^2 P$, can be assumed finite and yet it can be shown that at almost all points of P'' no kind of tangential plane exists. As a matter of fact, at almost all points x of P'' the upper density of the set P'' in any cone of vertex x is positive.

2. Denote by $c(a, r)$ and $S(a, r)$ respectively the circle and the sphere of centre a and of radius r. We have the theorem

THEOREM 1. *If $c(a_i, r_i)$ $(i = 1, 2, 3)$ are three coplanar circles such that the centre of no one of them is inside another one and*

$$r_1 \geqslant r_2 \geqslant r_3, \qquad a_1 a_3 \leqslant \tfrac{3}{2} r_3, \qquad a_2 a_3 \leqslant \tfrac{3}{2} r_3, \tag{1}$$

then the angle $a_1 a_3 a_2 > 38°$.

The proof is left to the reader.

COROLLARY 1. If $S(a_i, r_i)$ $(i = 1, 2, 3)$ are three spheres such that the centre of no one of them is inside another, and if the conditions (1) are satisfied, then the angle $a_1 a_3 a_2 > 38°$.

COROLLARY 2. Given a sphere $S(a, r)$ and a number of spheres $S(a_i, r_i)$ $(i = 1, 2, \ldots, k)$ satisfying the conditions

(i) the centre of no one of the spheres $S(a, r)$, $S(a_i, r_i)$ is inside another,

(ii) $\qquad r \leqslant r_i \quad \text{and} \quad a a_i \leqslant \tfrac{3}{2} r \quad (i = 1, 2, \ldots, k_1)$,

then $\qquad k_1 < \dfrac{2}{1 - \cos 19°}, \quad \text{i.e.} \quad k_1 \leqslant 36.$

THEOREM 2. *If $c(a_i, r_i)$ $(i = 1, 2, 3)$ are three coplanar circles, such that the centre of no one of them is inside another, the circles $c(a_1, r_1)$, $c(a_2, r_2)$ meet $c(a_3, r_3)$, and*

$$r_1 \geqslant r_2 \geqslant r_3, \qquad a_1 a_3 > \tfrac{3}{2} r_3, \qquad a_2 a_3 > \tfrac{3}{2} r_3,$$

then the angle $a_1 a_3 a_2 > 28°$.

The proof is left to the reader.

COROLLARY. Given a sphere $S(a, r)$ and a number of spheres $S(a_i, r_i)$ $(i = 1, 2, \ldots, k_2)$ satisfying the conditions

(i) the centre of no one of the spheres $S(a, r)$ and $S(a_i, r_i)$ is inside another,

(ii) each sphere $S(a_i, r_i)$ meets the sphere $S(a, r)$,

(iii) $\qquad r \leqslant r_i \quad \text{and} \quad a a_i > \tfrac{3}{2} r \quad (i = 1, 2, \ldots, k_2)$,

then $$k_2 < \frac{2}{1-\cos 14^\circ}, \quad \text{i.e.} \quad k_2 \leqslant 67.$$

THEOREM 3. *Given a sphere $S(a,r)$ and spheres $S(a_i,r_i)$ $(i=1,2,\ldots,k)$ satisfying the conditions*

(i) *the centre of no one of the spheres $S(a,r)$, $S(a_i,r_i)$ is inside another,*

(ii) *each sphere $S(a_i,r_i)$ meets the sphere $S(a,r)$,*

(iii) $$r_i \geqslant r$$

then $$k \leqslant 36+67 = 103.$$

The theorem follows immediately from Corollary 2 of Theorem 1 and the corollary of Theorem 2.

LEMMA 1. *Given a sequence of spheres*

$$S(a_i,r_i) \quad (i=1,2,\ldots)$$

satisfying the conditions

(i) $$r_1 \geqslant r_2 \geqslant r_3 \geqslant \ldots,$$

(ii) *the centre of no one of the spheres is inside another,*

then the sequence can be divided into 104 sub-sequences C_j $(j=1,2,\ldots,104)$ such that no pair of spheres belonging to the same sub-sequence have points in common.

Proof. First take $S(a_1,r_1)$ to C_1, $S(a_2,r_2)$ to $C_2,\ldots$, $S(a_{104},r_{104})$ to C_{104}. By Theorem 3 the sphere $S(a_{105},r_{105})$ cannot meet all the preceding spheres and, consequently, there is at least one C_j whose sphere does not meet $S(a_{105},r_{105})$. Then we take $S(a_{105},r_{105})$ to such C_j. After the first n $(\geqslant 105)$ spheres have been distributed between $C_1,\ldots, C_{104}$ we see again that, by Theorem 3, $S(a_{n+1},r_{n+1})$ cannot meet more than 103 of the preceding spheres and, consequently, there is at least one C_j whose spheres do not meet $S(a_{n+1},r_{n+1})$. Take it to such C_j. This proves that the required distribution of the spheres is possible.

Given a set E of points in the coordinate space, we denote by $P(E)$ the orthogonal projection of the set E on the plane $z=0$.

DEFINITION 1. Given an open plane set Q, consider all decompositions $Q = \sum Q_i$ of Q into a finite or an enumerable set of subsets. Then the number

$$l(Q) = \text{l. bd.} \sum dQ_i,$$

where dQ_i is the diameter of Q_i, is called *the linear extension of Q.*

DEFINITION 2. Linear extension of any plane set Q is defined as the linear extension of the set of all the interior points of Q.

DEFINITION 3. Linear extension of a three-dimensional set E is defined as the linear extension of its projection on the plane $z = 0$, i.e.

$$l(E) = l\{P(E)\}.$$

Denote by E the surface $z = f(x, y)$, where $f(x, y)$ is a continuous function defined in the rectangle R $[a \leqslant x \leqslant b,\ c \leqslant y \leqslant d]$. Take a sphere $S(\alpha, \rho)$ such that its projection $c\{\phi(\alpha), \rho\}$ is included in R.

LEMMA 2. *The set of all the points of E inside $S(\alpha, \rho)$ can be included in a set U of spheres such that $\sum_U r^2 < 4\rho l_0$, where $l_0 = l\{ES(\alpha, \rho)\}$ and $\sum_U r^2$ denotes the sum of squares of the radii of all the spheres of U.*

Proof. The set of all the points of E that are inside $S(\alpha, \rho)$ can be included in a set of vertical cylinders (not necessarily circular) of total width less than $l_0+\epsilon$ (meaning by width of a cylinder the diameter of its base), each one of height 2ρ. Any such cylinder of width λ can be included in a prism of base a square of side λ and height 2ρ, and any such prism can be included in a set of less than $4\rho/\lambda$ spheres of radius λ. For this set

$$\sum r^2 < \frac{4\rho}{\lambda}\lambda^2 = 4\rho\lambda,$$

and the lemma follows.

We shall now suppose that $f(x, y)$ is given in a rectangle R', $a' \leqslant x \leqslant b'$, $c' \leqslant y \leqslant d'$ $(a' < a < b < b',\ c' < c < d < d')$, including the rectangle R, and that it is absolutely continuous in the Tonelli sense: that is

(i) $f(x, y)$ is an absolutely continuous function of x for almost all values of y, and an absolutely continuous function of y for almost all values of x;

(ii) $$\iint_{R'} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\, dxdy < \infty.$$

Given a set G in space we shall denote by $K(G)$ the value of the integral

$$\iint_{P(EG)} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\, dxdy.$$

Remark. Note that $\iint_R \sqrt{(1+f_x^2)}\,dxdy$ is the integral with respect to y of the length of the section of the surface by a variable plane perpendicular to the Y-axis, a similar thing being true for the integral extended over any domain.

LEMMA 3. $\qquad K\{S(a,2\rho)\} > \rho l\{ES(a,\rho)\}.$

The proof follows from the following two propositions.

Prop. (i). $\qquad l(r) = l\{ES(a,r)\}$

is a non-decreasing function of r.

Prop. (ii). *For any* r, *and for* $r' > r$ *and sufficiently near* r

$$K\{S'(a,r')-S'(a,r)\} \geqslant (r'-r)l(r),$$

where $S'(a,r)$ *denotes the interior of the sphere* $S(a,r)$.

Prop. (i) is obvious; Prop. (ii) can be proved in the following way. The set of interior points of $P\{ES(a,r)\}$ consists of a finite or enumerable set of connected open domains, disjoint from one another. Taking the closure of each of them, we obtain a finite or an enumerable set of closed domains, disjoint from one another. (Of course it may happen that the closures of a number of disjoint open domains form one closed domain.) The sum of the diameters of all these closed domains is greater than or equal to $l(r)$. Take a finite number of these domains so that the sum of their diameters is greater than $l(r)/\sqrt{2}$. Let the domains be D^i ($i = 1, 2,..., k$) and their diameters be d^i ($i = 1, 2,..., k$). Let $\delta > 0$ be the shortest distance between a pair of points belonging to two different domains. Projection of D^i on at least one of the two axes OX, OY is greater than or equal to $d^i/\sqrt{2}$. Let it be the projection on OY. Denote it by (y_1^i, y_2^i) so that $y_2^i - y_1^i \geqslant d^i/\sqrt{2}$. For any y' ($y_1^i \leqslant y' \leqslant y_2^i$) denote by $x_1^i(y')$, $x_2^i(y')$ the smallest and the largest values of x for which $(x,y') \in D^i$. Take now r' such that $0 < r'-r < \frac{1}{2}\delta$ and define the numbers $x_1'^i(y') < x_1^i(y')$ and $x_2'^i(y') > x_2^i(y')$ so that the arc of the curve $z = f(x,y')$ (y' constant) from $x_1'^i(y')$ to $x_1^i(y')$ and the arc from $x_2^i(y')$ to $x_2'^i(y')$ are each of length equal to $r'-r$. Each such arc lies inside the sphere $S(a,r')$ except possibly its end point, and no point of it lies inside $S(a,r)$. Arcs corresponding to different D^i do not meet, because the length of each arc is less than $\frac{1}{2}\delta$ while the distance between any pair of domains is not less than δ. Denote by D'^i the domain D^i increased by the intervals

$$\{x_1'^i(y'), x_1^i(y')\} \quad \text{and} \quad \{x_2^i(y'), x_2'^i(y')\}$$

on any line $y = y'$, not including the beginning of the first one and the end of the second one. By the definition of $x_1'^i(y')$, $x_2'^i(y')$,

$$\int_{x_1'^i(y')}^{x_2'^i(y')} \sqrt{(1+f_x^2)}\,dx = \int_{x_1^i(y')}^{x_2^i(y')} \sqrt{(1+f_x^2)}\,dx = r'-r.$$

Consequently,

$$\iint_{D'^i-D^i} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\,dxdy > 2(r'-r)(y_2^i-y_1^i)$$
$$> \sqrt{2}(r'-r)dD^i.$$

The sets D'^i-D^i $(i = 1, 2,..., k)$ do not overlap and all are included in $P\{ES'(a,r')-ES'(a,r)\} = P(r',r)$. Therefore

$$K\{S'(a,r')-S'(a,r)\} = \iint_{P(r',r)} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\,dxdy$$
$$> \iint_{\sum_{i=1}^{k}(D'^i-D^i)} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\,dxdy$$
$$> (r'-r)\sqrt{2}\sum_{i=1}^{k} dD^i$$
$$> (r'-r)l(r).$$

This completes the proof.

LEMMA 4. *The set of the points a of E at which*

$$\overline{\lim}\,\frac{K\{S(a,r)\}}{r^2} < \frac{1}{72}$$

is of Carathéodory two-dimensional measure zero.

As before, the surface E' is defined by the equation $z = f(x,y)$ in a rectangle R' including the rectangle R in its interior. Our proof refers to the part E of E' corresponding to the rectangle R, but it is convenient to have the surface extended beyond R.

Take a sequence of positive numbers $\epsilon_1 > \epsilon_2 > ..., \epsilon_n \to 0$. Denote by $U(\delta, E)$ any set of spheres, of diameter less than δ, including the whole of E. We assume that the projections of all the spheres are within R'. Write

$$\text{l. bd.} \sum_{U(2^{-k},E)} \pi r^2 = \Gamma_k \quad (0 < \Gamma_k < \infty)$$

and take a $U(2^{-k}, E) = U^k$ such that

$$\sum_{U^k} \pi r^2 < \Gamma_k + \tfrac{1}{2}\epsilon_k. \tag{1}$$

Write $U^k = U_1^k + U_2^k$, where any sphere $S(a,r)$ of U^k belongs to U_1^k

or to U_2^k according as $l\{ES(a,r)\} \geqslant \frac{1}{8}r$ or $< \frac{1}{8}r$. If a sphere $S(a,r)$ belongs to U_2^k, then by Lemma 2 the set of points of $ES(a,r)$ can be included in a set V of spheres $S(a',r')$ such that

$$\sum_V r'^2 < \tfrac{1}{2}r^2.$$

Replace every sphere of U_2^k by such a set V of spheres. Then the set U_2^k of spheres is replaced by a set $U_2'^k$ such that

$$EU_2^k \subset U_2'^k,$$

and

$$\sum_{U_2'^k} \pi r^2 < \tfrac{1}{2} \sum_{U_2^k} \pi r^2. \tag{2}$$

We have $U_1^k + U_2'^k \supset E$ and consequently

$$\sum_{U_1^k} \pi r^2 + \sum_{U_2'^k} \pi r^2 \geqslant \Gamma_k. \tag{3}$$

By (1), (2), (3),

$$\sum_{U_2^k} \pi r^2 < \epsilon_k.$$

Let

$$\overline{U} = \varlimsup U_2^k.$$

We have $\Lambda^2 \overline{U} = 0$ and, writing $E\overline{U} = E_0$,

$$\Lambda^2 E_0 = 0.$$

Let now a be an arbitrary point of $E - E_0$. It belongs to an infinite number of sets U_1^k. Consequently there exists an infinite sequence $S(a_n, r_n)$ of spheres including a and such that

$$r_n \to 0, \quad \text{and} \quad l\{ES(a_n, r_n)\} > \tfrac{1}{8}r_n.$$

By Lemma 3,

$$K\{S(a_n, 2r_n)\} > \tfrac{1}{8}r_n^2.$$

Now $S(a, 3r_n) \supset S(a_n, 2r_n)$ and consequently

$$K\{S(a, 3r_n)\} > \tfrac{1}{8}r_n^2 = \tfrac{1}{72}(3r_n)^2, \tag{4}$$

and thus

$$\varlimsup \frac{K\{S(a,r)\}}{r^2} \geqslant \tfrac{1}{72}.$$

This completes the proof.

Remark. If (4) is satisfied at a point a, then

$$\frac{K\{S(a, 2^{-n})\}}{2^{-2n}} > \frac{1}{4 \times 72} - \epsilon > \frac{1}{300} \tag{5}$$

for infinitely many integral values of n, and consequently this holds for almost all points of E.

LEMMA 5. *If H ($\subset R$) is an open set and E_H the part of E projected on H, then*

$$\Lambda^2 E_H < 10^5 K(E_H).^* \tag{6}$$

* $\Lambda^2 E$ is defined as $\varliminf \frac{1}{4}\pi \sum (dA_i)^2$ for all decompositions $A = \sum A_i$, and $dA_i \to 0$.

Proof. Denote by E_0 the set of the points of E_H at which (4) is not satisfied, so that $\Lambda^2 E_0 = 0$. Given $\epsilon > 0$ and $\rho > 0$, include all the points of E_0 in a set U_0 of spheres of radii $< \frac{1}{2}\rho$ and such that

$$\sum_{U_0} \pi r^2 < \epsilon. \tag{7}$$

To any point $a \in (E_H - E_0)$ correspond an infinite number of positive integers n for which the following conditions are satisfied

$$2^{-n} < \tfrac{1}{2}\rho, \qquad P\{S(a, 2^{-n})\} \subset H, \qquad \frac{K\{S(a, 2^{-n})\}}{2^{-2n}} > \frac{1}{300}. \tag{8}$$

Denote by E^ν the set of all those points of $E_H - E_0$ for which the smallest value of n satisfying (8) is ν.

Then
$$E_H - E_0 = \sum_{\nu_0+1}^{\infty} E^\nu.$$

Let a_1 be an arbitrary point of E^{ν_0+1}; construct the sphere $S_1 = S(a_1, 2^{-\nu_0-1})$. Let a_2 be a point of E^{ν_0+1} outside or on the surface of S_1; define the sphere $S_2 = S(a_2, 2^{-\nu_0-1})$. Let a_3 be a point of E^{ν_0+1} not inside one of the spheres S_1, S_2; define the sphere $S_3 = S(a_3, 2^{-\nu_0-1})$ and so on. In this way all the points of E^{ν_0+1} will be included in a finite number of spheres, say i_1. Let a_{i_1+1}, then, be an arbitrary point of E^{ν_0+2} that is not inside one of the spheres $S_1, S_2, \ldots, S_{i_1}$. Define the sphere $S_{i_1+1} = S(a_{i_1+1}, 2^{-\nu_0-2})$, and so on. In this way all the points of $E_H - E_0$ will be included in the sequence U_1 of spheres $S_1, S_2, \ldots$.

The set of spheres $U_0 + U_1$ includes all the points of E_H. Now the spheres of the sequence U_1 satisfy the conditions of Lemma 1, and therefore the sequence U_1 can be represented as the sum

$$U_1 = \sum_{k=1}^{104} C_k$$

of 104 sub-sequences C_k such that no pair of spheres belonging to the same sub-sequence have points in common. As each sphere $S(a, r)$ of C_k satisfies (8), that is

$$r^2 < 300 K\{S(a, r)\},$$

we have
$$\sum_{C_k} \pi r^2 < 300\pi \sum_{C_k} K\{S(a, r)\}.$$

But as spheres of C_k have no points in common

$$\sum_{C_k} K\{S(a, r)\} = K(C_k) \leqslant K(E_H).$$

Hence

$$\sum_{U_0+U_1} \pi r^2 = \sum_{U_0} \pi r^2 + \sum_{k=1}^{104} \sum_{U_k} \pi r^2 < \epsilon + 104.300\pi K(E_H).$$

As this is true for any ϵ, ρ, we conclude that

$$\Lambda^2 E_H < 10^5 K(E_H),$$

and the proof is complete.

THEOREM 4. *For any Λ^2 measurable set W in space,*

$$\Lambda^2 W \geqslant |P(W)|.$$

The theorem follows in an obvious way from the definition of $\Lambda^2 W$.

THEOREM 5. *Given a circular cone A, all of whose generators form with the Z-axis an angle $> \theta_0 > 0$, then there exists a finite constant $K = K(\theta_0)$ such that for any set B on A whose projection $P(B)$ is an open set,*

$$\Lambda^2 B < K|P(B)|.$$

The theorem is an immediate corollary of the fact that $\Lambda^2 B$ in this case coincides with the value of the area of B in the usual sense.

Given a plane t, not parallel to the Z-axis, a point a on t, and a positive number r, we shall denote by $[t, a, r]$ the vertical cylinder through the circle in the plane t, whose centre is a and radius r.

THEOREM 6. *Let a be a point of E and t be the generalized tangential plane at a. If t is not parallel to the Z-axis, then*

$$\lim_{r \to 0} \frac{\Lambda^2\{E[t, a, r]\}}{\pi r^2} \geqslant 1.$$

Proof. Let n be the normal to t at a. Take the circular cone, whose vertex is a and whose generators form an angle $(\frac{1}{2}\pi - \alpha)$with n. Denote by A the part of the surface of the cone inside $[t, a, r]$ and write $A = A_1 + A_2$, where A_1 is the part of the surface of the cone that lies above the plane t, and A_2 the part that lies below t. Denote further by B the part of A_1 that lies below E and the part of A_2 that lies above E. By the definition of the generalized tangential plane

$$\lim_{r \to 0} \frac{|P(B)|}{\pi r^2} = 0 \tag{8'}$$

for any $\alpha > 0$, and, by Theorem 5,

$$\lim_{r \to 0} \frac{\Lambda^2 B}{\pi r^2} = 0. \tag{9}$$

Denote by $P(G,t)$ the orthogonal projection of G on t, while $P(G)$ denotes the orthogonal projection of G on $z=0$. Writing

$$W = E[t,a,r]+B \tag{10}$$

it can easily be seen that

$$P(W,t) \supset P(A_1,t)P(A_2,t). \tag{11}$$

Given $\eta > 0$, as small as we please, there exists $\alpha_0 > 0$ such that, for any $\alpha < \alpha_0$,

$$|P(A_1,t)P(A_2,t)| > (1-\eta)\pi r^2. \tag{12}$$

We assume α to be less than α_0, and keeping it constant a number $r_0 > 0$ can be defined, such that for any $r < r_0$

$$\frac{\Lambda^2 B}{\pi r^2} < \eta. \tag{13}$$

By Theorem 4, $\qquad \Lambda^2 W \geqslant |P(W,t)|,$

and, by (11) and (12),

$$\Lambda^2 W \geqslant (1-\eta)\pi r^2.$$

By (10) and (13),

$$\Lambda^2\{E[t,a,r]\} > (1-2\eta)\pi r^2$$

for any $r < r_0$, which proves the proposition.

THEOREM 7. *If $G \subset R$ is an open set and E_G is the part of the surface E whose projection on $z=0$ is G, then*

$$\Lambda^2 E_G = \iint_G \sqrt{(1+f_x^2+f_y^2)}\, dxdy.$$

Let η be an arbitrarily small positive number. Since

$$\iint_{R'} \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\, dxdy < \infty, \tag{14}$$

there exists a positive number ϵ such that, for any measurable set $H \subset R'$,

$$10^5 \iint_H \{\sqrt{(1+f_x^2)}+\sqrt{(1+f_y^2)}\}\, dxdy < \eta \tag{15}$$

whenever $|H| < \epsilon$.

Let $G_1 \subset G$ be a measurable set satisfying the following conditions:

(i) every point of G_1 is a density point of G_1;

(ii) at every point of G_1 the generalized tangential plane to E_G exists;

(iii) the functions f_x, f_y are continuous and bounded on G_1;

(iv) $|G_1| > |G| - \frac{1}{2}\epsilon$.

For every point a of E_{G_1} we shall consider, just as before, the generalized tangential plane t, the cylinder $[t,a,r]$, the cone $A = A_1+A_2$ whose axis is the normal to t at a and whose generators form an angle $\frac{1}{2}\pi-\alpha$ with the axis, and B will have the same meaning as before. Denote further by $e(\bar{a},r)$, where $\bar{a}$ is the projection of a, the base of the cylinder $[t,a,r]$, i.e. the projection of the circle $tS(a,r)$, so that $e(\bar{a},r)$ is an ellipse whose centre is $\bar{a}$. By (iii) there exists a positive number m such that, for all points of G_1, the ratio of the smaller axis of $e(\bar{a},r)$ to the larger is greater than m. We have from the definition of $e(\bar{a},r)$

$$\sqrt{(1+f_x^2+f_y^2)_a}\,|e(\bar{a},r)| = \pi r^2. \tag{16}$$

With every point $\bar{a}$ of G_1 we associate ellipses $e(\bar{a},r)$, satisfying the following conditions:

(i+) $e(\bar{a},r) \subset G$ and $r < \frac{1}{3}\rho$, where ρ is a fixed positive number as small as we please;

(ii+) $\operatorname{osc}\{\sqrt{(1+f_x^2+f_y^2)}, G_1 e(\bar{a},r)\} < \eta$, where the expression on the left-hand side denotes the oscillation of $\sqrt{(1+f_x^2+f_y^2)}$ on $G_1 e(\bar{a},r)$;

(iii+) $$|G_1 e(\bar{a},r)| > \frac{|e(\bar{a},r)|}{1+\eta};$$

(iv+) $$S\{a,r(1+\eta)\} \supset A_1+A_2 = A;$$

(v+) $$|P(B)| < \frac{\epsilon}{3|G|}\,|e(\bar{a},r)|;$$

(vi+) $$\Lambda^2\{E[t,a,r]\} > (1-\eta)\pi r^2.$$

By the definition of G_1, (i+), (ii+), (iii+) are obviously satisfied for sufficiently small r, and (iv+) is satisfied if the angle $\frac{1}{2}\pi-\alpha$ between the generators and the axis of the cone A is sufficiently near $\frac{1}{2}\pi$. It follows from (8′) that (v+) is satisfied for sufficiently small values of r, and from Theorem 6 that (vi+) is satisfied for sufficiently small values of r. The set of all the ellipses defined in this way cover the set G_1 in the Vitali sense. Therefore a set Q of non-overlapping ellipses $e(\bar{a},r)$ can be chosen covering almost all points of G_1. We have

$$|G_1| \leqslant |Q| \leqslant |G|. \tag{17}$$

Denote by T the set of the cylinders $[t,a,r]$ corresponding to all the ellipses of Q. We have

$$\Lambda^2 E_G \geqslant \sum_T \Lambda^2\{E[t,a,r]\}.$$

By (vi+), $$\Lambda^2 E_G \geqslant (1-\eta)\sum_Q \pi r^2, \tag{18}$$

where $\sum_Q \pi r^2$ denotes the sum of values of πr^2 for the values of r corresponding to all the ellipses of Q. It follows from (16), (ii+), (iii+), (15) that, for every ellipse $e(\bar{a}, r)$ of Q, we have (when θ is a number between -1 and $+1$ and may have different values)

$$\begin{aligned}\pi r^2 &= \sqrt{(1+f_x^2+f_y^2)_a}\,|e(\bar{a}, r)| \\ &= \sqrt{(1+f_x^2+f_y^2)_a}\,|G_1 e(\bar{a}, r)|(1+\theta\eta) \\ &= (1+\theta\eta)\iint_{G_1 e(\bar{a},r)} \{\sqrt{(1+f_x^2+f_y^2)}+\theta\eta\}\,dxdy \\ &= (1+3\theta\eta)\iint_{G_1 e(\bar{a},r)} \sqrt{(1+f_x^2+f_y^2)}\,dxdy,\end{aligned}$$

and consequently

$$\sum_Q \pi r^2 = (1+3\theta\eta)\iint_G \sqrt{(1+f_x^2+f_y^2)}\,dxdy+\theta\eta. \tag{19}$$

By (18), $$\Lambda^2 E_G > (1-4\eta)\iint_G \sqrt{(1+f_x^2+f_y^2)}\,dxdy-\eta,$$

and, as this holds for any $\eta > 0$,

$$\Lambda^2 E_G \geqslant \iint_G \sqrt{(1+f_x^2+f_y^2)}\,dxdy. \tag{20}$$

We shall complete the proof by establishing an inequality of the opposite sense.

Take any $e(\bar{a}, r) \in Q$ and the corresponding surface $E[t, a, r]$. By (iv+) the sphere $S\{a, r(1+\eta)\}$ includes all the points of E corresponding to the set $e(\bar{a}, r)-P(B)$. Denote by $\mathfrak{S}$ the set of the spheres $S\{a, r(1+\eta)\}$ corresponding to all the ellipses of Q. Denoting by d the diameter of every sphere of $\mathfrak{S}$, we have

$$\sum_{\mathfrak{S}} \tfrac{1}{4}\pi d^2 = (1+\eta)^2 \sum_Q \pi r^2. \tag{21}$$

The set $\mathfrak{S}$ contains all the points of E corresponding to the points of the set

$$G_2 = Q-\sum_Q P(B).$$

By (iv), (17), (v+), $$|G_2| > |G|-\epsilon.$$

Writing $$G = G_2+G_3$$

we have $$|G_3| < \epsilon.$$

Including G_3 in an open set G_3' such that $|G_3'| < \epsilon$, we conclude

from (15) and from Lemma 5 that the set E_{G_η} can be included in a set $\mathfrak{S}'$ of spheres of radii $\frac{1}{4}\rho$, such that

$$\sum_{\mathfrak{S}'} \tfrac{1}{4}\pi d^2 < \eta. \tag{22}$$

Thus the whole of the set E_G is included in the set of spheres $\mathfrak{S}+\mathfrak{S}'$, whose diameters are less than ρ.

By (19), (21), (22),

$$\sum_{\mathfrak{S}+\mathfrak{S}'} \tfrac{1}{4}\pi d^2 = (1+6\theta\eta)\iint_G \sqrt{(1+f_x^2+f_y^2)}\,dxdy+3\theta\eta.$$

As this is true for any ρ and η, we have

$$\Lambda^2 E_G \leqslant \iint_G \sqrt{(1+f_x^2+f_y^2)}\,dxdy, \tag{23}$$

and the theorem follows from (20) and (23).

The theorem can be immediately generalized to the case of any measurable set $G \subset R$.

[*Added 15 June 1945.*] Since I sent this paper for publication two papers by H. Federer on measure of a surface have appeared.* The author modifies Carathéodory's definition of measure and proves that in the Tonelli case the measure thus defined is equal to the area. However, his measure is of an isolated nature, that is, it is not fitted into a continuous scale of measures of various dimensions and it is not clear whether it can be so fitted. Such a fitting should be a very essential feature of measure, since otherwise it is only very particular classes of sets that can be compared from the point of view of measure, all the other sets having measure equal either to 0 or to ∞. Federer's measure definitely cannot be fitted into the Hausdorff scale since a surface can be constructed whose Federer's measure is finite, while the Hausdorff α-dimensional measure is equal to ∞ for all $\alpha < 3$.

* *Trans. American Math. Soc.* 55 (1944).

ON EXPANSIONS IN EIGENFUNCTIONS (VII)

By E. C. TITCHMARSH (*Oxford*)

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1. In a previous paper* I gave a proof of the eigenfunction expansion associated with the differential equation

$$\frac{d^2y}{dx^2}-\{q(x)-\lambda\}y=0 \quad (0\leqslant x<\infty). \tag{1.1}$$

The function $f(x)$ to be expanded was subject to rather heavy restrictions including the existence of $f'(x)$ and $f''(x)$. The result was that the expansion was not merely convergent for any particular x, but absolutely and uniformly convergent over any finite interval.

In the present paper I consider the situation when $f(x)$ is not so restricted. In particular cases, such as the Hermite case, the expansion is valid under the same conditions as an ordinary Fourier series. It seems difficult to prove this more generally without imposing heavy restrictions on the function $q(x)$. On the other hand, there is a quite general theorem involving summability of a certain type. It will be proved that,

if $q(x)$ is any function which tends steadily to infinity, and $f(x)$ is $L^2(0,\infty)$, and the eigenvalues, eigenfunctions, and 'Fourier coefficients' are λ_n, $\psi_n(x)$, and c_n, then

$$\lim_{\nu\to\infty}\sum_{n=0}^{\infty}\frac{\nu}{\nu+\lambda_n}c_n\psi_n(x)=f(x) \tag{1.2}$$

almost everywhere, and wherever $f(x)$ is continuous.

2. Before proving this theorem, the opportunity is taken of indicating some improvements which could be made in the previous argument. In the first place, the notation there used led to the eigenvalues being negative in ordinary cases. To remedy this we now take as variable a λ equivalent to the previous $-w$. Also the fundamental assumption of the previous paper (§ 5) was that the function $\Psi_+(x, w)$ had only simple poles at points independent of x. A simpler assumption, which leads to the result in a more natural way, is that the function there called $l_1(w)$ is meromorphic.

* E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 12 (1941), 33–50.

In its revised form the argument of the previous paper would begin as follows.

Let $\phi(x) = \phi(x, \lambda)$, $\theta(x) = \theta(x, \lambda)$ be the solutions of (1.1) such that

$$\left.\begin{aligned} \phi(0) &= \sin\alpha, & \phi'(0) &= -\cos\alpha \\ \theta(0) &= \cos\alpha, & \theta'(0) &= \sin\alpha \end{aligned}\right\}, \tag{2.1}$$

where α is real. It is known that these are integral functions of λ. We have

$$W_x(\phi, \theta) = W_0(\phi, \theta) = \sin^2\alpha + \cos^2\alpha = 1.$$

The general solution of (1.1) is

$$\theta(x) + l\phi(x),$$

where l is any real or complex number. We now consider those solutions which satisfy a real boundary condition at $x = b$, say

$$[\theta(b) + l\phi(b)]\cos\beta + [\theta'(b) + l\phi'(b)]\sin\beta = 0,$$

where β is real. For each b, as $\cot\beta$ varies, l describes a circle in the complex plane, say C_b. We then prove, as in the previous paper, that, if λ is not real, the circles C_b converge as $b \to \infty$ either to a limit-circle or to a limit-point. If m denotes the limit-point, or any point on the limit-circle, $\theta(x) + m\phi(x)$ is $L^2(0, \infty)$, and

$$\int_0^\infty |\theta(x) + m\phi(x)|^2 \, dx \leqslant -\frac{\mathbf{I}(m)}{v},$$

where $\lambda = u + iv$.

It is clear from the formulae of § 2 of the previous paper that $l(\lambda) = l(\lambda, b, \beta)$ is an analytic function of λ, regular except on the real axis. Now on the circle C_b (if $v > 0$)

$$|l|^2 \int_0^b |\phi|^2 \, dx - \int_0^b |\theta|^2 \, dx \leqslant \int_0^b |\theta + l\phi|^2 \, dx = -\frac{\mathbf{I}(l)}{v} \leqslant \frac{|l|}{v}.$$

Hence

$$|l| < K(\lambda, v, b).$$

But the argument about the circles C_b shows that, for a given λ the region of the l-plane covered by the circles C_b decreases as b increases. Hence $l(\lambda)$ converges boundedly in any region entirely in the upper (or lower) half of the λ-plane. Hence its limit $m(\lambda)$ is analytic in either half-plane.

We now assume that *the only singularities of* $m(\lambda)$ *are poles*, say at $\lambda_0, \lambda_1, \ldots$.

Let $$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda). \tag{2.2}$$

If $f(x)$ is any function of $L^2(0,\infty)$, let

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy+\phi(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy. \tag{2.3}$$

As in § 3 of the previous paper, we have

$$\int_0^\infty |\Phi(x,\lambda)|^2\,dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\,dx. \tag{2.4}$$

It follows from this that *the poles of* $m(\lambda)$ *are all simple*. Take

$$f(x) = \begin{cases} 1 & (0 < x < \xi \leqslant 1), \\ 0 & (x > \xi). \end{cases}$$

Then, for $1 \leqslant x \leqslant 2$,

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^\xi \phi(y,\lambda)\,dy.$$

Hence (2.4) gives

$$\int_1^2 |\psi(x,\lambda)|^2\,dx\left|\int_0^\xi \phi(y,\lambda)\,dy\right|^2 \leqslant \frac{1}{v^2}.$$

As λ tends to a pole λ_n, the left-hand side is asymptotic to

$$|m(\lambda)|^2\int_1^2 |\phi(x,\lambda_n)|^2\,dx\left|\int_0^\xi \phi(y,\lambda_n)\,dy\right|^2.$$

We can clearly choose ξ so that the last integral does not vanish, and we should then obtain a contradiction if $m(\lambda)$ had a pole of order higher than the first. Let $m(\lambda)$ have the residue r_n at λ_n.

3. We have—e.g. by (2.2) of the previous paper—if λ and λ' are not real,

$$(\lambda'-\lambda)\int_0^b \psi(x,\lambda)\psi(x,\lambda')\,dx = W_0\{\psi(x,\lambda)\psi(x,\lambda')\}-W_b\{\psi(x,\lambda),\psi(x,\lambda')\}. \tag{3.1}$$

From (2.1) we obtain

$$W_0\{\psi(x,\lambda),\,\psi(x,\lambda')\} = m(\lambda)-m(\lambda').$$

Making $b \to \infty$, the second term on the right of (3.1) tends to 0, by (2.13) of the previous paper. Hence, making $b \to \infty$,

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda')\,dx = \frac{m(\lambda)-m(\lambda')}{\lambda'-\lambda}. \tag{3.2}$$

In particular, if $\lambda' = \bar{\lambda}$,

$$\int_0^\infty |\psi(x,\lambda)|^2\,dx = -\frac{\mathrm{I}\{m(\lambda)\}}{v}. \tag{3.3}$$

Now let λ_n be an eigenvalue, i.e. a pole of $m(\lambda)$, and let $\lambda' = \lambda_n + iv$. Then for any fixed X, as $v \to 0$,

$$\int_0^X |v\psi(x,\lambda') + ir_n\phi(x,\lambda_n)|^2\,dx \to 0.$$

For the left-hand side is

$$\int_0^X |v\theta(x,\lambda') + \{vm(\lambda') + ir_n\}\phi(x,\lambda') - ir_n\{\phi(x,\lambda') - \phi(x,\lambda_n)\}|^2\,dx$$

$$\leqslant v^2\int_0^X |\theta(x,\lambda')|^2\,dx + |vm(\lambda') + ir_n|^2\int_0^X |\phi(x,\lambda')|^2\,dx +$$

$$+ r_n^2\int_0^X |\phi(x,\lambda') - \phi(x,\lambda_n)|^2\,dx,$$

and each term here clearly tends to 0. Also, by (3.3),

$$\int_0^\infty |v\psi(x,\lambda')|^2\,dx \leqslant |vm(\lambda')|,$$

which is bounded as $v \to 0$, since the pole of $m(\lambda)$ at λ_n is simple.

On multiplying (3.2) by iv/r_n and making $v \to 0$, it therefore follows from Lemma γ of the previous paper that $\phi(x,\lambda_n)$ is L^2, and

$$\int_0^\infty \psi(x,\lambda)\phi(x,\lambda_n)\,dx = \frac{1}{\lambda-\lambda_n}. \tag{3.4}$$

If λ tends to a different pole λ_m, on multiplying this by iv/r_m and making $v \to 0$, we obtain

$$\int_0^\infty \phi(x,\lambda_m)\phi(x,\lambda_n)\,dx = 0. \tag{3.5}$$

If λ tends to the same pole λ_n, we have

$$\frac{iv}{r_n(\lambda-\lambda_n)} \to \frac{1}{r_n}$$

and we obtain

$$\int_0^\infty |\phi(x,\lambda_n)|^2\,dx = \frac{1}{r_n}. \tag{3.6}$$

Hence the functions $\psi_n(x) = r_n^{\frac{1}{2}}\phi(x,\lambda_n)$ form a normal orthogonal set over $(0,\infty)$.

4. Let $f(x)$ be any function of $L^2(0,\infty)$, and define $\Phi(x,\lambda)$ by (2.3). Since $\psi(x,\lambda)$ is L^2, the second integral in (2.3) converges, and in fact converges uniformly in any finite region entirely in the upper (or lower) half-plane. Hence $\Phi(x,\lambda)$ is analytic in either half-plane.

Let $X > x$, and

$$\Phi_X(x,\lambda) = \psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy + \phi(x,\lambda)\int_x^X \psi(y,\lambda)f(y)\,dy.$$

This function is clearly regular except at the points λ_n, where it has simple poles with residues

$$r_n\phi(x,\lambda_n)\int_0^X \phi(y,\lambda_n)f(y)\,dy.$$

Now

$$|\Phi_X(x,\lambda)-\Phi_{X'}(x,\lambda)| = \left|\phi(x,\lambda)\int_X^{X'}\psi(y,\lambda)f(y)\,dy\right|$$

$$\leqslant |\phi(x,\lambda)|\left\{\int_X^{X'}|\psi(y,\lambda)|^2\,dy\int_X^{X'}|f(y)|^2\,dy\right\}^{\frac{1}{2}} \leqslant \frac{\delta}{v}$$

by (3.3), where $\delta \to 0$ as X and $X' \to \infty$, for a fixed x, uniformly for $|\lambda| \leqslant \Lambda$. Using Lemma α of the previous paper, it follows that $\Phi_X(x,\lambda)$ converges uniformly, as $X \to \infty$, in any finite region not containing any of the poles λ_n. Hence $\Phi(x,\lambda)$ is a single analytic function, regular except at the λ_n. The residue of $\Phi(x,\lambda)$ at λ_n is the limit of that of $\Phi_X(x,\lambda)$, and so is

$$r_n\phi(x,\lambda_n)\int_0^\infty \phi(y,\lambda_n)f(y)\,dy = c_n\psi_n(x),$$

where

$$c_n = \int_0^\infty \psi_n(y)f(y)\,dy.$$

We can now deduce as in the previous paper the following convergence theorem:

Let $f(x)$ and $f''(x)-q(x)f(x)$ belong to $L^2(0,\infty)$, and let $f(x)$ satisfy the boundary conditions

$$f(0)\cos\alpha+f'(0)\sin\alpha = 0, \tag{4.1}$$

$$\lim_{x\to\infty} W\{\psi(x,\lambda), f(x)\} = 0. \tag{4.2}$$

Then

$$f(x) = \sum_{n=0}^{\infty} c_n\psi_n(x) \quad (0<x<\infty), \tag{4.3}$$

the series being absolutely and uniformly convergent in any finite interval.

In the previous paper this was deduced from the corresponding theorem on mean convergence; actually it is easy to prove it directly in a similar way.

We next prove that, *if λ is not equal to any of the eigenvalues λ_m, and $\psi(x)$ is any function of $L^2(0,\infty)$,*

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n\psi_n(x)}{\lambda-\lambda_n}. \tag{4.4}$$

If λ is not real, this follows from the above theorem, as in § 8 of the previous paper. Also

$$\sum_{n=0}^{\infty} \left|\frac{\psi_n(x)}{\lambda_n-\lambda}\right|^2$$

is convergent for every complex λ, and hence also is uniformly convergent in any finite region of the λ-plane not including any of the points λ_n. Since $\sum c_n$ is convergent, it follows that the series on the right of (4.4) is also uniformly convergent in any finite region not including any of the λ_n. It therefore follows from continuity that (4.4) holds if λ has any real value other than the λ_n.

5. We now come to the main object of the paper, the summability theorem, and begin by proving a series of lemmas.

LEMMA α. *Let y be a solution of*

$$\frac{d^2y}{dx^2} = p(x)y$$

where $p(x)>0$ for $x\geqslant x_0$. Let $y(x)$ belong to $L^2(0,\infty)$. Then y and dy/dx have opposite signs, and tend steadily to zero as $x\to\infty$.

Suppose, for example, that $y(x_0) > 0$. Then $y'(x_0) < 0$, or y would tend steadily to infinity, since it is convex downwards where it is positive. Thus $y'(x) < 0$ as long as $y(x) > 0$. But $y(x)$ cannot change sign; for, if x_1 were its first zero, and $y(x) < 0$ for x just greater than x_1, there would be a point $x_1+\delta$ such that

$$y(x_1+\delta) < 0, \qquad y'(x_1+\delta) < 0.$$

Hence y would decrease steadily past this point, and so tend to $-\infty$ (being convex upwards). This is impossible if $y(x)$ is $L^2(x_0, \infty)$.

It follows that, for $x \geqslant x_0$, $y(x) > 0$ and $y'(x) < 0$. Hence y decreases steadily, and so tends to a limit, which must be zero. Since $y''(x) > 0$, $y'(x)$ increases steadily, and so tends to a limit, which clearly must also be zero.

6. LEMMA β. *Let the conditions of Lemma α be satisfied, and suppose also that $p(x)$ is non-decreasing. Then, if $y(x_0) > 0$,*

$$y(x) \leqslant y(x_0)\exp\Big[-\int_{x_0}^{x} \{p(\xi)\}^{\frac{1}{2}}\, d\xi\Big] \quad (x > x_0). \tag{6.1}$$

We have

$$-y'(x)y''(x) = p(x)y(x)\{-y'(x)\} \geqslant p(x_0)y(x)\{-y'(x)\} \quad (x \geqslant x_0).$$

Integrating from x_0 to x_1

$$\tfrac{1}{2}\{y'^2(x_0)-y'^2(x_1)\} \geqslant \tfrac{1}{2}p(x_0)\{y^2(x_0)-y^2(x_1)\}.$$

Making $x_1 \to \infty$, and replacing x_0 by x, it follows that

$$y'^2(x) \geqslant p(x)y^2(x).$$

Hence

$$-\frac{y'(x)}{y(x)} \geqslant \{p(x)\}^{\frac{1}{2}}. \tag{6.2}$$

Integrating from x_0 to x,

$$\log y(x_0) - \log y(x) \geqslant \int_{x_0}^{x} \{p(\xi)\}^{\frac{1}{2}}\, d\xi,$$

and (6.1) follows.

7. LEMMA γ. *Let the conditions of Lemma β be satisfied for $x \geqslant 0$, and let*

$$y(0)\cos\alpha + y'(0)\sin\alpha = 1 \tag{7.1}$$

where $\sin\alpha \neq 0$. Then, if $p(0) > \operatorname{cosec}^2\alpha$,

$$|y(x)| \leqslant \frac{\exp\Big[-\int_{0}^{x} \{p(\xi)\}^{\frac{1}{2}}\, d\xi\Big]}{\{p(0)\}^{\frac{1}{2}}|\sin\alpha| - 1}. \tag{7.2}$$

We have $|y'(0)\sin\alpha| \leqslant 1+|y(0)| \leqslant 1+\{p(0)\}^{-\frac{1}{2}}|y'(0)|$,

by (6.2). Hence $$|y'(0)| \leqslant \frac{1}{|\sin\alpha|-\{p(0)\}^{-\frac{1}{2}}}.$$

Also, by (6.1) and (6.2),

$$|y(x)| \leqslant |y(0)|\exp\Big[-\int_0^x \{p(\xi)\}^{\frac{1}{2}}\,d\xi\Big]$$

$$\leqslant |y'(0)|\{p(0)\}^{-\frac{1}{2}}\exp\Big[-\int_0^x \{p(\xi)\}^{\frac{1}{2}}\,d\xi\Big],$$

and (7.2) follows.

8. Lemma δ. *Under the same conditions*

$$|y(x)| \geqslant \frac{2y(x_0)\exp[-(x-x_0)\{p(x+1)\}^{\frac{1}{2}}]}{1+\{1+1/p(x+1)\}^{\frac{1}{2}}}. \tag{8.1}$$

Suppose for example that $y(x)$ is positive. Since $p(x)$ is non-decreasing,

$$-y'(x)y''(x) = p(x)y(x)\{-y'(x)\} \leqslant p(x_1)y(x)\{-y'(x)\} \quad (x \leqslant x).$$

Integrating from x to x_1,

$$\tfrac{1}{2}\{y'^2(x)-y'^2(x_1)\} \leqslant \tfrac{1}{2}p(x_1)\{y^2(x)-y^2(x_1)\}$$

$$\leqslant \tfrac{1}{2}p(x_1)y^2(x).$$

Writing temporarily $a = \{p(x_1)\}^{\frac{1}{2}}$, $b = -y'(x_1)$, this gives

$$\left(\frac{dy}{dx}\right)^2 \leqslant a^2y^2+b^2.$$

Hence $$\int_y^{y_0} \frac{dy}{\sqrt{(a^2y^2+b^2)}} \leqslant x-x_0,$$

where $y_0 = y(x_0)$. Hence

$$\frac{1}{a}\log\frac{y_0+\sqrt{(y_0^2+b^2/a^2)}}{y+\sqrt{(y^2+b^2/a^2)}} \leqslant x-x_0,$$

$$y+\sqrt{\left(y^2+\frac{b^2}{a^2}\right)} \geqslant \left\{y_0+\sqrt{\left(y_0^2+\frac{b^2}{a^2}\right)}\right\}e^{-a(x-x_0)} \geqslant 2y_0\,e^{-a(x-x_0)}. \tag{8.2}$$

Also, since y is positive and convex downwards,

$$|y'(x)| \leqslant \frac{y(x')-y(x)}{x-x'} \quad (0 < x' < x).$$

Taking $x' = x-1$, this gives

$$|y'(x)| \leqslant y(x-1).$$

Hence, taking $x_1 = x+1$ in (8.2),

$$y(x)+\left\{y^2(x)+\frac{y^2(x)}{p(x+1)}\right\}^{\frac{1}{2}} \geqslant 2y_0\, e^{-\{p(x+1)\}^{\frac{1}{2}}(x-x_0)},$$

and (8.1) follows.

9. Lemma ϵ. *Let y satisfy the above conditions, where now $p(x) = t^2+q(x)$ and $q(x)$ is non-decreasing and independent of t. Then, if $y(0)$ is given, as $t \to \infty$,*

$$y(x) \sim y(0)e^{-xt} \tag{9.1}$$

uniformly in any fixed range of values of x.

This follows at once from Lemmas β and δ.

Lemma ζ. *Under the same conditions, if (7.1) holds (instead of $y(0)$ being given)*

$$y(x) \sim -\frac{e^{-xt}}{t\sin\alpha}. \tag{9.2}$$

It follows from (6.2), with $x = 0$, that $|y'(0)/y(0)|$ is large for large t, and hence from (7.1) that $y'(0)$ has the same sign as $\sin\alpha$, if t is large enough. Hence $y(x)$ has the opposite sign to $\sin\alpha$. Thus the sign in (9.2) is correct. It will then be sufficient to consider one case, e.g. that in which $y(x)$ is positive.

The required upper bound for $y(x)$ follows at once from (7.2). Also (7.1) gives

$$1 \leqslant |y(0)|+|y'(0)\sin\alpha| \leqslant \{p(0)\}^{-\frac{1}{2}}|y'(0)|+|y'(0)\sin\alpha|$$

by (6.2) with $x = 0$. Hence

$$|y'(0)| \geqslant \frac{1}{\sin\alpha+\{p(0)\}^{-\frac{1}{2}}}. \tag{9.3}$$

Also, if $x_1 > x$,

$$\begin{aligned} y(x_1)-y(x) &= \int_x^{x_1} y'(\xi)\,d\xi \\ &= [-y'(\xi)(x_1-\xi)]_x^{x_1} + \int_x^{x_1} y''(\xi)(x_1-\xi)\,d\xi \\ &= y'(x)(x_1-x)+\int_x^{x_1} p(\xi)y(\xi)(x_1-\xi)\,d\xi, \end{aligned}$$

$$y'(x) = \frac{y(x_1)-y(x)}{x_1-x} - \frac{1}{x_1-x}\int_x^{x_1} p(\xi)y(\xi)(x_1-\xi)\,d\xi,$$

and hence

$$|y'(x)| \leqslant \frac{y(x)-y(x_1)}{x_1-x}+(x_1-x)p(x_1)y(x).$$

Now, by (8.1),

$$y(x_1) \geqslant y(x)e^{-(x_1-x)t}\{1+O(1/t)\}$$

if x and x_1 lie in a fixed interval. Hence

$$|y'(x)| \leqslant y(x)\left[\frac{1-e^{-(x_1-x)t}}{x_1-x}+O\left\{\frac{1}{(x_1-x)t}\right\}+O\{(x_1-x)p(x_1)\}\right]$$

Taking $(x_1-x)t$ small and $(x_1-x)t^2$ large, e.g. $x_1 = x+t^{-\frac{1}{2}}$, we obtain

$$|y'(x)| \leqslant ty(x)(1+\delta) \quad \{t > t_0(\delta)\}. \tag{9.4}$$

The lower bound for $y(x)$ implied by (9.2) now follows from (8.1), (9.4) and (9.3).

10. LEMMA η. *Let $\phi(x, -t^2)$ be the solution of*

$$\frac{d^2\phi}{dx^2} = \{q(x)+t^2\}\phi(x)$$

such that $\phi(0) = \sin\alpha$, $\phi'(0) = -\cos\alpha$. Then, as $t \to \infty$,

$$\phi(x) = \begin{cases} \cosh xt \sin\alpha + O\left(\dfrac{e^{xt}}{t}\right) & (\alpha \neq 0), \qquad (10.1) \\ -\dfrac{\sinh xt}{t} + O\left(\dfrac{e^{xt}}{t^2}\right) & (\alpha = 0) \qquad (10.2) \end{cases}$$

uniformly in any fixed x-interval.

Formulae of this type have been given by many writers. We can prove (10.1) by taking $u(x) = \phi(x)\operatorname{cosec}\alpha$ and replacing ρ by it in the argument given by Ince (*Ordinary Differential Equations*, § 11.4). We can prove (10.2) similarly.

11. THEOREM. *Let $f(x)$ be $L^2(0, \infty)$, and let $q(x)$ tend steadily to infinity with x. Then* (1.2) *holds for every value of x for which*

$$\int_0^\eta |f(x+y)-f(x)|\, dy = o(\eta) \tag{11.1}$$

as $\eta \to 0$. In particular (i) *it holds almost everywhere,* (ii) *it holds wherever $f(x)$ is continuous.*

In view of (4.4), it is sufficient to prove that, if (11.1) holds, then

$$\Phi(x, -t^2) \sim -\frac{f(x)}{t^2}. \tag{11.2}$$

Now

$$\Phi(x,-t^2) = \psi(x,-t^2)\int_0^x \phi(y,-t^2)f(y)\,dy + \\ +\phi(x,-t^2)\int_x^\infty \psi(y,-t^2)f(y)\,dy.$$

By (2.1) and (2.2)

$$\psi(0,-t^2) = \cos\alpha + m(-t^2)\sin\alpha,$$
$$\psi'(0,-t^2) = \sin\alpha - m(-t^2)\cos\alpha,$$

so that

$$\psi(0,-t^2)\cos\alpha + \psi'(0,-t^2)\sin\alpha = 1. \qquad (11.3)$$

Suppose first that $\alpha \neq 0$. Then $\phi(x,-t^2)$ satisfies (10.1), and $\psi(x,-t^2)$ satisfies (9.2), uniformly in any finite range. Also, by (7.2),

$$\psi(x,-t^2) = O\left[t^{-1}\exp\left\{-\int_0^x \{t^2+q(\xi)\}^{\frac{1}{2}}\,d\xi\right\}\right] \\ = O(e^{-xt})$$

for all x and sufficiently large t. Hence, if $\delta > 0$,

$$\psi(x,-t^2)\int_0^{x-\delta} \phi(y,-t^2)f(y)\,dy = O\left\{\frac{e^{-xt}}{t}\int_0^{x-\delta} e^{yt}|f(y)|\,dy\right\} = O(e^{-\delta t}),$$

and

$$\phi(x,-t^2)\int_{x+\delta}^\infty \psi(y,-t^2)\psi(y)\,dy \\ = O\left[e^{xt}\left\{\int_{x+\delta}^\infty e^{-2yt}\,dy\int_0^\infty f^2(y)\,dy\right\}^{\frac{1}{2}}\right] = O(e^{-\delta t}).$$

Hence

$$\Phi(x,-t^2) = \psi(x,-t^2)\int_{x-\delta}^x \phi(y,-t^2)f(y)\,dy + \\ +\phi(x,-t^2)\int_x^{x+\delta} \psi(y,-t^2)f(y)\,dy + O(e^{-\delta t})$$
$$= f(x)\left\{\psi(x,-t^2)\int_{x-\delta}^x \phi(y,-t^2)\,dy + \phi(x,-t^2)\int_x^{x+\delta} \psi(y,-t^2)\,dy\right\} + \\ +\psi(x,-t^2)\int_{x-\delta}^x \phi(y,-t^2)\{f(y)-f(x)\}\,dy + \\ +\phi(x,-t^2)\int_x^{x+\delta} \psi(y,-t^2)\{f(y)-f(x)\}\,dy + O(e^{-\delta t}). \qquad (11.4)$$

The coefficient of $f(x)$ is asymptotic to

$$-\frac{e^{-xt}}{t\sin\alpha}\int_{x-\delta}^{x}\tfrac{1}{2}e^{yt}\sin\alpha\,dy-\tfrac{1}{2}e^{xt}\sin\alpha\int_{x}^{x+\delta}\frac{e^{-yt}}{t\sin\alpha}\,dy=-\frac{1}{t^2}+O(e^{-\delta t}).$$

Let
$$\chi(\eta)=\int_0^{\eta}|f(x+y)-f(x)|\,dy.$$

Then the third term on the right of (11.4) is

$$O\left[e^{xt}\int_{x}^{x+\delta}\frac{e^{-yt}}{t}|f(y)-f(x)|\,dy\right]$$

$$=O\left[\frac{1}{t}\int_0^{\delta}e^{-ut}|f(x+u)-f(x)|\,du\right]$$

$$=O\left[\frac{1}{t}[e^{-ut}\chi(u)]_0^{\delta}+\int_0^{\delta}e^{-ut}\chi(u)\,du\right]$$

$$=O(e^{-\delta t})+o\left(\int_0^{\delta}e^{-ut}u\,du\right)$$

$$=O(e^{-\delta t})+o\left(\frac{1}{t^2}\right).$$

Similarly for the second term. Altogether we obtain

$$\Phi(x,-t^2)=-\frac{f(x)}{t^2}+o\left(\frac{1}{t^2}\right).$$

A similar argument holds in the case $\alpha=0$, by using (9.1) and (10.2). This proves the theorem.

ON EXPANSIONS IN EIGENFUNCTIONS (VIII)

By E. C. TITCHMARSH (*Oxford*)

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1. LET $q(x)$ be a function of x which increases steadily to infinity in $(0, \infty)$, and let $\psi_0(x)$, $\psi_1(x)$,... be the eigenfunctions of the differential equation

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y = 0, \tag{1.1}$$

which satisfy a given boundary condition at $x = 0$. Let λ_0, λ_1,... be the eigenvalues, and

$$c_n = \int_0^\infty f(x)\psi_n(x)\,dx,$$

the 'Fourier coefficients' of a function $f(x)$ of $L^2(0, \infty)$.

It was proved in the previous paper* that the series $\sum c_n\psi_n(x)$ is summable almost everywhere to the sum $f(x)$, in the sense that

$$\lim_{r\to\infty}\sum_{n=0}^{\infty}\frac{r}{r+\lambda_n}c_n\psi_n(x) = f(x). \tag{1.2}$$

In this paper I consider conditions under which the series converges in the ordinary sense. I now assume a good deal more about the function $q(x)$, but the conditions are satisfied in all ordinary cases.

Let $q(x)$ be twice differentiable, and

$$q'(x) > 0, \qquad q''(x) > 0 \qquad (x > 0). \tag{1.3}$$

Let

$$q''(x) = O\left[\{q'(x)\}^{\frac{14}{13}}\right]. \tag{1.4}$$

The condition (1.4) implies a certain regularity in the growth of the function $q(x)$, but does not restrict its rate of growth. It also gives

$$\begin{aligned} q'(x) &= \int_0^x q''(t)\,dt+q'(0) \\ &= O\left[\int_0^x \{q'(t)\}^{\frac{14}{13}}\,dt\right] \\ &= O\left[\{q'(x)\}^{\frac{1}{13}}\int_0^x q'(t)\,dt\right] \\ &= O\left[\{q'(x)\}^{\frac{1}{13}}q(x)\right], \\ q'(x) &= O\left[\{q(x)\}^{\frac{14}{13}}\right]. \end{aligned} \tag{1.5}$$

* See above, pp. 103–14.

The results are as follows.

THEOREM 1. *Let $f(y)$ be $L^2(0,\infty)$, and $f(y)=0$ in an interval of which x is an interior point. Then*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) = 0.$$

THEOREM 2. *Let $f(y)$ be of bounded variation over an interval including the point x, and zero outside this interval. Then*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Combining the results, we clearly obtain

THEOREM 3. *Let $f(y)$ be $L^2(0,\infty)$, and let it be of bounded variation in an interval including the point x. Then*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

2. I begin by proving some lemmas on the eigenfunctions $\psi_n(x)$. Let h_n be defined by $q(h_n)=\lambda_n$. We have

$$\psi_n''(x)+\{\lambda_n-q(x)\}\psi_n(x)=0. \tag{2.1}$$

For $x<h_n$, the coefficient of $\psi_n(x)$ in (2.1) is positive. Hence $\psi_n(x)$ is concave downwards where it is positive, and upwards where it is negative. Hence it has just one maximum or minimum between consecutive zeros.

LEMMA α. *The successive maxima of $|\psi_n(x)|$ increase steadily in $0<x<h_n$.*

Let
$$\omega(x)=\psi_n^2(x)+\frac{\psi_n'^2(x)}{\lambda_n-q(x)}.$$

Then
$$\omega'(x)=\frac{\psi_n'^2(x)q'(x)}{\{\lambda_n-q(x)\}^2}\geqslant 0.$$

Hence $\omega(x)$ is non-decreasing, and increasing except at isolated points. At a maximum of $|\psi_n(x)|$, $\psi_n'(x)=0$, so that $\omega(x)=\psi_n^2(x)$. The lemma clearly follows from this.*

LEMMA β. *Let $\eta_1, \eta_2, \ldots$ be the zeros of $\psi_n(x)$. Then*

$$\frac{\psi_n'(\eta_\nu)}{|\lambda_n-q(\eta_\nu)|^{\frac{1}{4}}}$$

increases steadily with η_ν in $(0, h_n)$.

* See G. Szegö, *Orthogonal Polynomials* (American Math. Soc. Colloquium Publications, XXIII), 161.

Since $$\omega(x) = \frac{\psi_n'^2(x)}{\lambda_n - q(x)}$$

at $x = \eta_1, \eta_2, \ldots$, this also follows from the above argument.

LEMMA γ. *For any fixed x (or x in a fixed interval)*

$$\psi_n'(x) = O(\lambda_n^{\frac{1}{2}} h_n^{-\frac{1}{2}}).$$

Let the succesive maxima and zeros of $|\psi_n(x)|$ in the interval $0 < x \leqslant \frac{1}{2}h_n$ be

$$\xi_1 < \eta_1 < \xi_2 < \ldots < \eta_{n'}.$$

Consider a particular η_μ. By Lemma β,

$$\frac{|\psi_n'(\eta_\mu)|}{|\lambda_n - q(\eta_\mu)|^{\frac{1}{2}}} \leqslant \frac{1}{n'-\mu+1} \sum_{\nu=\mu}^{n'} \frac{|\psi_n'(\eta_\nu)|}{|\lambda_n - q(\eta_\nu)|^{\frac{1}{2}}}.$$

Since $q(x)$ is convex downwards,

$$q(\tfrac{1}{2}x) \leqslant \tfrac{1}{2}\{q(x)+q(0)\} \leqslant \tfrac{3}{4}q(x)$$

if x is large enough. Hence, if n is large enough,

$$q(\tfrac{1}{2}h_n) \leqslant \tfrac{3}{4}q(h_n) = \tfrac{3}{4}\lambda_n.$$

Hence $$|\psi_n'(\eta_\mu)| \leqslant \frac{2}{n'-\mu+1} \sum_{\nu=\mu}^{n'} |\psi_n'(\eta_\nu)|. \tag{2.2}$$

Suppose, for example, that $\psi_n(x)$ is positive between ξ_ν and η_ν. Then

$$\int_{\xi_\nu}^{\eta_\nu} \psi_n(x)\,dx = \int_{\xi_\nu}^{\eta_\nu} \frac{-\psi_n''(x)}{\lambda_n - q(x)}\,dx$$

$$\geqslant \frac{1}{\lambda_n} \int_{\xi_\nu}^{\eta_\nu} \{-\psi_n''(x)\}\,dx = -\psi_n'(\eta_\nu)/\lambda_n.$$

Hence $$|\psi_n'(\eta_\nu)| \leqslant \lambda_n \int_{\xi_\nu}^{\eta_\nu} |\psi_n(x)|\,dx. \tag{2.3}$$

Thus (2.2) gives

$$|\psi_n'(\eta_\mu)| \leqslant \frac{2\lambda_n}{n'-\mu+1} \int_0^{\frac{1}{2}h_n} |\psi_n(x)|\,dx$$

$$\leqslant \frac{2\lambda_n}{n'-\mu+1} \left[\int_0^{\frac{1}{2}h_n} \{\psi_n(x)\}^2\,dx \int_0^{\frac{1}{2}h_n} dx \right]^{\frac{1}{2}}$$

$$\leqslant \frac{2^{\frac{1}{2}}\lambda_n h_n^{\frac{1}{2}}}{n'-\mu+1}. \tag{2.4}$$

Now let x be a given positive number. Let ξ_μ, $\xi_{\mu+1}$ be the nearest zeros of $\psi_n'(y)$ below and above x, and η_μ the zero of $\psi_n(y)$ between ξ_μ and $\xi_{\mu+1}$. Then $|\psi_n'(x)| \leqslant |\psi_n'(\eta_\mu)|$. Also it is clear from the argument of my paper 'On the eigenvalues of differential equations'* that

$$n'-\mu \geqslant \frac{1}{\pi}\int_x^{\frac{1}{2}h_n} \{\lambda_n - q(y)\}^{\frac{1}{2}}\,dy - O(1)$$

$$\geqslant \frac{1}{\pi}\int_x^{\frac{1}{2}h_n} (\tfrac{1}{2}\lambda_n)^{\frac{1}{2}}\,dy - O(1)$$

$$> A h_n \lambda_n^{\frac{1}{2}}. \tag{2.5}$$

Hence
$$\psi_n'(\eta_\mu) = O(\lambda_n^{\frac{1}{2}} h_n^{-\frac{1}{2}}) \tag{2.6}$$

and the result follows. The extension to an interval of values of x is obvious.

LEMMA 8. *For any fixed x (or x in a fixed interval)*

$$\psi_n(x) = O(h_n^{-\frac{1}{2}}).$$

Consider again an interval (ξ_ν, η_ν) where

$$\psi_n(x) > 0, \qquad \psi_n'(x) < 0, \qquad x \leqslant \tfrac{1}{2}h_n.$$

Then

$$\psi_n'(x)\psi_n''(x) = \{\lambda_n - q(x)\}\psi_n(x)\{-\psi_n'(x)\}$$

$$\geqslant \tfrac{1}{2}\lambda_n \psi_n(x)\{-\psi_n'(x)\}.$$

Hence

$$\int_{\xi_\nu}^{\eta_\nu} \psi_n'(x)\psi_n''(x)\,dx \geqslant \tfrac{1}{2}\lambda_n \int_{\xi_\nu}^{\eta_\nu} \psi_n(x)\{-\psi_n'(x)\}\,dx,$$

$$\psi_n'^2(\eta_\nu) \geqslant \tfrac{1}{2}\lambda_n \psi_n^2(\xi_\nu).$$

Hence, by (2.6), $\qquad \psi_n(\xi_\mu) = O(h_n^{-\frac{1}{2}}),$

and the result follows as before.

It is easily verified that, in particular cases, these results are the best possible. For example, in the Hermite case,

$$q(x) = x^2, \qquad \lambda_n = 2n+1, \qquad h_n = (2n+1)^{\frac{1}{2}},$$

* *J. of London Math. Soc.* 19 (1944), 66–8.

so that Lemma δ gives $\psi_n(x) = O(n^{-\frac{1}{4}})$. Actually

$$\psi_n(x) = \frac{e^{-\frac{1}{2}x^2}H_n(x)}{(n!\,2^n\pi^{\frac{1}{2}})^{\frac{1}{2}}},$$

and,* for a fixed x, $\quad H_n(x) = O\left\{\dfrac{n!}{\Gamma(\frac{1}{2}n+1)}\right\}$.

The same result follows from these formulae.

3. Let $N(x)+1$ be the number of eigenvalues not exceeding x, so that

$$N(x) = n \quad (\lambda_{n-1} < x \leqslant \lambda_n).$$

Let $h(x)$ be the inverse function of $q(x)$, so that $h(\lambda_n) = h_n$. Then

$$N(\lambda) \geqslant \frac{1}{\pi}\int_0^{h(\lambda)} \{\lambda - q(x)\}^{\frac{1}{2}}\,dx - O(1). \tag{3.1}$$

For $\lambda = \lambda_n$, this is (2.3) of my last quoted paper.† For $\lambda_{n-1} < \lambda \leqslant \lambda_n$, if we denote the integral by $I(\lambda)$,

$$N(\lambda) \geqslant N(\lambda_n) - 1 \geqslant I(\lambda_n)/\pi - O(1) \geqslant I(\lambda)/\pi - O(1)$$

since $I(\lambda)$ is steadily increasing.

LEMMA ϵ.

$$N(\lambda) < \frac{1}{\pi}\int_0^{h(\lambda)} \{\lambda + \sqrt{\lambda} - q(x)\}^{\frac{1}{2}}\,dx + O(1).$$

It is sufficient to prove this for each λ_n. Consider separately the intervals $0 \leqslant x \leqslant h_n - \delta$ and $h_n - \delta < x \leqslant h_n$. In the former, compare the number of zeros of $\psi_n(x)$ with the number of zeros of

$$\{\lambda_n + \sqrt{\lambda_n} - q(x)\}^{-\frac{1}{4}}\cos\left[\int_0^x \{\lambda_n + \sqrt{\lambda_n} - q(t)\}^{\frac{1}{2}}\,dt\right].$$

This function has a zero between any two consecutive zeros of $\psi_n(x)$ if

$$\frac{5q'^2(x)}{16\{\lambda_n + \sqrt{\lambda_n} - q(x)\}^2} + \frac{q''(x)}{4\{\lambda_n + \sqrt{\lambda_n} - q(x)\}} < \lambda_n^{\frac{1}{2}}$$

for $0 < x \leqslant h_n - \delta$. This is true if

$$\frac{5q'^2(h_n-\delta)}{16\{\lambda_n - q(h_n-\delta)\}^2} + \frac{q''(h_n-\delta)}{4\{\lambda_n - q(h_n-\delta)\}} < \lambda_n^{\frac{1}{2}}.$$

* Szegö, loc. cit., 94.

† *J. of London Math. Soc.* 19 (1944), 66–8.

Now $$\lambda_n - q(h_n-\delta) = \int_{h_n-\delta}^{h_n} q'(t)\,dt \geqslant \delta q'(h_n-\delta).$$

Hence we want $$\frac{5}{16\delta^2} + \frac{q''(h_n-\delta)}{4\delta q'(h_n-\delta)} < \lambda_n^{\frac{1}{2}}.$$

Let $\delta = \lambda_n^{-\frac{1}{2}}$. Then we want

$$\frac{q''(h_n-\delta)}{4\lambda_n^{-\frac{1}{2}} q'(h_n-\delta)} < \tfrac{11}{16}\lambda_n^{\frac{1}{2}},$$

i.e. $$q''(h_n-\delta) < \tfrac{11}{4}\{q(h_n)\}^{\frac{1}{2}} q'(h_n-\delta).$$

This is true for sufficiently large h_n if

$$q''(x) < \tfrac{11}{4}\{q(x)\}^{\frac{1}{2}} q'(x)$$

for sufficiently large x; and this follows from (1.4) and (1.5).

Hence the number of zeros of $\psi_n(x)$ in $(0,\ h_n-\delta)$ does not exceed

$$\frac{1}{\pi}\int_0^{h_n-\delta} \{\lambda_n + \sqrt{\lambda_n - q(t)}\}^{\frac{1}{2}}\,dt + O(1).$$

In the interval $(h_n-\delta, h_n)$

$$\lambda_n - q(x) = \int_x^{h_n} q'(t)\,dt \leqslant (h_n-x)q'(h_n).$$

Hence $\psi_n(x)$ has a zero between any two consecutive zeros of a solution of

$$\frac{d^2z}{dx^2} + (h_n-x)q'(h_n)z = 0 \quad (h_n-\delta \leqslant x \leqslant h_n).$$

Put $h_n - x = \{q'(h_n)\}^{-\frac{1}{3}}\xi$. Then

$$\frac{d^2z}{d\xi^2} + \xi z = 0 \quad (0 \leqslant \xi \leqslant \delta\{q'(h_n)\}^{\frac{1}{3}}).$$

A solution of this is

$$\xi^{\frac{1}{2}} J_{\frac{1}{3}}(\tfrac{2}{3}\xi^{\frac{3}{2}}) \sim \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \xi^{-\frac{1}{4}} \cos(\tfrac{2}{3}\xi^{\frac{3}{2}} - \tfrac{5}{12}\pi).$$

Hence the number of zeros does not exceed

$$\frac{2}{3\pi}[\delta\{q'(h_n)\}^{\frac{1}{2}}]^{\frac{3}{2}} + O(1) = \frac{1}{\pi}\int_0^{\delta} \{xq'(h_n)\}^{\frac{1}{2}}\,dx + O(1).$$

Also

$$\int_0^{\delta} [\{xq'(h_n)\}^{\frac{1}{2}} - \{\lambda_n - q(h_n - x)\}^{\frac{1}{2}}]\, dx$$

$$= O \int_0^{\delta} \frac{xq'(h_n) - \lambda_n + q(h_n - x)}{\{xq'(h_n)\}^{\frac{1}{2}}}\, dx$$

$$= O \int_0^{\delta} \frac{x^2 q''(h_n)}{\{xq'(h_n)\}^{\frac{1}{2}}}\, dx = O\left[\frac{\delta^{\frac{1}{2}} q''(h_n)}{\{q'(h_n)\}^{\frac{1}{2}}}\right]$$

$$= O[q''(h_n)\{q(h_n)\}^{-\frac{1}{4}}\{q'(h_n)^{-\frac{1}{4}}].$$

Using (1.4) and (1.5), this is

$$O\left[\{q(h_n)\}^{-\frac{5}{6}}\{q'(h_n)\}^{\frac{1}{2}\frac{5}{6}}\right] = O(1).$$

Hence the number of zeros of $\psi_n(x)$ in $(h_n - \delta, h_n)$ does not exceed

$$\frac{1}{\pi} \int_{h_n - \delta}^{h_n} \{\lambda_n - q(x)\}^{\frac{1}{2}}\, dx + O(1),$$

and the lemma follows.

LEMMA ζ. $\qquad N(\lambda + \sqrt{\lambda}) - N(\lambda) = O\{h(\lambda)\}.$

By Lemma ϵ,

$$N(\lambda + \sqrt{\lambda}) < \frac{1}{\pi} \int_0^{h(\lambda + \sqrt{\lambda})} \{\lambda + 3\sqrt{\lambda} - q(x)\}^{\frac{1}{2}}\, dx + O(1)$$

for λ large enough. Hence

$$\pi\{N(\lambda + \sqrt{\lambda}) - N(\lambda)\} < \int_0^{h(\frac{1}{2}\lambda)} + \int_{h(\frac{1}{2}\lambda)}^{h(\lambda)} [\{\lambda + 3\sqrt{\lambda} - q(x)\}^{\frac{1}{2}} - \{\lambda - q(x)\}^{\frac{1}{2}}]\, dx +$$

$$+ \int_{h(\lambda)}^{h(\lambda + \sqrt{\lambda})} \{\lambda + 3\sqrt{\lambda} - q(x)\}^{\frac{1}{2}}\, dx + O(1) = I_1 + I_2 + I_3 + O(1).$$

Now

$$\{\lambda + 3\sqrt{\lambda} - q(x)\}^{\frac{1}{2}} - \{\lambda - q(x)\}^{\frac{1}{2}} = \frac{3\sqrt{\lambda}}{\{\lambda + 3\sqrt{\lambda} - q(x)\}^{\frac{1}{2}} + \{\lambda - q(x)\}^{\frac{1}{2}}}.$$

For $x \leqslant h(\frac{1}{2}\lambda)$, $q(x) \leqslant \frac{1}{2}\lambda$. Hence

$$I_1 = \int_0^{h(\frac{1}{2}\lambda)} O(1)\, dx = O\{h(\tfrac{1}{2}\lambda)\} = O\{h(\lambda)\}.$$

Also

$$I_2 = O\int_{h(\frac{1}{2}\lambda)}^{h(\lambda)} \frac{\sqrt{\lambda}}{\{\lambda+3\sqrt{\lambda}-q(x)\}^{\frac{1}{2}}}\,dx$$

$$= O\frac{\sqrt{\lambda}}{q'\{h(\frac{1}{2}\lambda)\}}\int_{h(\frac{1}{2}\lambda)}^{h(\lambda)} \frac{q'(x)}{\{\lambda+3\sqrt{\lambda}-q(x)\}^{\frac{1}{2}}}\,dx$$

$$= O\frac{\sqrt{\lambda}}{q'\{h(\frac{1}{2}\lambda)\}}[\{\lambda+3\sqrt{\lambda}-q(x)\}^{\frac{1}{2}}]_{h(\frac{1}{2}\lambda)}^{h(\lambda)} = O\left[\frac{\lambda}{q'\{h(\frac{1}{2}\lambda)\}}\right].$$

Now $$q(x)-q(0) = \int_0^x q'(t)\,dt < xq'(x).$$

Hence $$q'\{h(\tfrac{1}{2}\lambda)\} > \frac{q\{h(\frac{1}{2}\lambda)\}-q(0)}{h(\frac{1}{2}\lambda)} = \frac{\frac{1}{2}\lambda-q(0)}{h(\frac{1}{2}\lambda)}.$$

Thus $$I_2 = O[h(\tfrac{1}{2}\lambda)] = O\{h(\lambda)\}.$$

Finally

$$I_3 = O[\lambda^{\frac{1}{2}}\{h(\lambda+\sqrt{\lambda})-h(\lambda)\}] = O\{\lambda^{\frac{1}{2}}h'(\lambda)\}$$

$$= O\left[\frac{\lambda^{\frac{1}{2}}}{q'\{h(\lambda)\}}\right] = O\left\{\frac{h(\lambda)}{\lambda^{\frac{1}{2}}}\right\} = O\{h(\lambda)\}.$$

This proves the lemma.

LEMMA η. *For* $\sqrt{\lambda} \leqslant \mu \leqslant \lambda$,

$$N(\lambda+\mu)-N(\lambda) = O\{\mu\lambda^{-\frac{1}{2}}h(\lambda)\}.$$

This follows on applying the previous lemma $O(\mu\lambda^{-\frac{1}{2}})$ times, and observing that, since $h(x)$ is concave downwards, $h(x) \leqslant 2h(\frac{1}{2}x)$.

LEMMA θ.

$$N(\lambda+\mu) = N(\lambda)\left\{1+O\left(\frac{\mu}{\lambda}\right)\right\} \quad (\sqrt{\lambda} \leqslant \mu \leqslant \lambda).$$

By (3.1)

$$N(\lambda) \geqslant \frac{1}{\pi}\int_0^{h(\frac{1}{2}\lambda)} (\tfrac{1}{2}\lambda)^{\frac{1}{2}}\,dx - O(1) > A\lambda^{\frac{1}{2}}h(\tfrac{1}{2}\lambda) > A\lambda^{\frac{1}{2}}h(\lambda),$$

and the result follows from the previous lemma.

4. We now proceed to the proof of Theorem 1. We have

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n\psi_n(x)}{\lambda-\lambda_n}. \tag{4.1}$$

We have to prove that, under the conditions of the theorem,

$$\lim_{|\lambda|=R} \int \Phi(x,\lambda)\, d\lambda = 0 \tag{4.2}$$

as $R \to \infty$ through values not equal to any of the eigenvalues.

Let $f(y) = 0$ for $x-\delta \leqslant y \leqslant x+\delta$. Then it follows from the analysis of the previous paper—in particular from (11.4)—that

$$\Phi(x,-r) = O(e^{-\delta\sqrt{r}}), \tag{4.3}$$

where r is real and positive, and tends to infinity.

Let $\lambda = u+iv$, and consider first the order of $\Phi(x,\lambda)$ on the parabola $v^2 = u$. Since $f(x)$ is $L^2(0,\infty)$, $\sum c_n^2$ is convergent. Hence

$$|\Phi(x,\lambda)| \leqslant \left\{\sum_{n=0}^{\infty} c_n^2 \sum_{n=0}^{\infty} \left|\frac{\psi_n(x)}{\lambda-\lambda_n}\right|^2\right\}^{\frac{1}{2}} = O\left\{\sum_{n=0}^{\infty} \frac{h_n^{-1}}{(\lambda_n-u)^2+v^2}\right\}^{\frac{1}{2}} \tag{4.4}$$

by Lemma δ. Now

$$\sum_{n=0}^{\infty} \frac{h_n^{-1}}{(\lambda_n-u)^2+v^2} \leqslant \sum_{\lambda_n \leqslant \frac{1}{2}u} \frac{h_n^{-1}}{(\frac{1}{2}u)^2} + \sum_{\frac{1}{2}u<\lambda_n\leqslant u-v} \frac{h_n^{-1}}{(u-\lambda_n)^2} + \\ + \sum_{u-v<\lambda_n\leqslant u+v} \frac{h_n^{-1}}{v^2} + \sum_{u+v<\lambda_n} \frac{h_n^{-1}}{(\lambda_n-u)^2}.$$

$$= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4, \text{ say.}$$

Here $$\Sigma_3 \leqslant \frac{N(u+v)-N(u-v)}{v^2 h(u-v)} = O\left(\frac{1}{v^2}\right) = O\left(\frac{1}{u}\right)$$

by Lemma η. Similarly,

$$\sum_{u+2^{k-1}v<\lambda_n\leqslant u+2^k v} \frac{h_n^{-1}}{(\lambda_n-u)^2} \leqslant \frac{N(u+2^k v)-N(u+2^{k-1}v)}{2^{2k-2}v^2 h(u+2^{k-1}v)}$$

$$= O\left\{\frac{1}{2^{k-1}v(u+2^{k-1}v)^{\frac{1}{2}}}\right\} = O\left(\frac{1}{2^{k-1}u}\right).$$

Summing with respect to k, we obtain

$$\Sigma_4 = O\left(\frac{1}{u}\right).$$

Similarly,

$$\Sigma_2 = O\left(\frac{1}{u}\right).$$

Also

$$\sum_{\frac{1}{4}u<\lambda_n\leqslant\frac{1}{2}u} \frac{1}{h_n u^2} = O\left(\frac{N(\frac{1}{2}u)-N(\frac{1}{4}u)}{h(\frac{1}{4}u)u^2}\right) = O(u^{-\frac{1}{2}}).$$

Adding $O(\log u)$ terms of the type, we get

$$\Sigma_1 = O(u^{-\frac{1}{2}}\log u) = O(1/u).$$

Hence $$\Phi(x,\lambda) = O(u^{-\frac{1}{4}}) \tag{4.5}$$

on the parabola.

This also holds throughout the part of the first quadrant above the parabola, since the right-hand side of (4.4) is a decreasing function of v, for a fixed u. Also, if u is replaced by $-u$, (4.4) gives

$$\Phi(x,\lambda) = O\Big(\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2 h_n}\Big) = O(1),$$

the convergence of the series following from the above argument. It follows that $\Phi(x,\lambda)$ is bounded in the region between the negative real axis and the upper half of the parabola $v^2 = u$.

We now apply the Phragmén-Lindelöf theorem. Let

$$F(\lambda) = \lambda^{\frac{1}{4}}\exp(-\tfrac{1}{2}i\delta\lambda^{\frac{1}{4}})\Phi(x,\lambda),$$

where $\lambda^{\frac{1}{4}}$ is positive on the positive real axis. By (4.3), $F(\lambda)$ is bounded on the negative real axis. On the parabola $v^2 = u$

$$\exp(-\tfrac{1}{2}i\delta\lambda^{\frac{1}{4}}) = \exp\Big\{-\tfrac{1}{2}i\delta u^{\frac{1}{4}}\Big(1+\frac{iv}{2u}+\ldots\Big)\Big\} = O(1).$$

Hence, by (4.5), $F(\lambda)$ is bounded on the parabola. In the region between,

$$F(\lambda) = O(|\lambda^{\frac{1}{4}}|e^{\frac{1}{4}\delta|\lambda|^{\frac{1}{4}}}) = O(e^{\delta|\lambda|^{\frac{1}{4}}}).$$

Hence $F(\lambda)$ is bounded in the region between, i.e. (putting $\lambda = re^{i\theta}$)

$$\Phi(x,\lambda) = O(r^{-\frac{1}{4}}e^{-\frac{1}{2}\delta r^{\frac{1}{4}}\sin\frac{1}{4}\theta}) = O(r^{-\frac{1}{4}}e^{-(\delta/2\pi)r^{\frac{1}{4}}\theta}).$$

Hence, if $Re^{i\eta}$ lies in the above region, i.e. if $R\sin^2\eta \geqslant \cos\eta$,

$$\int_{|\lambda|=R,\ \eta\leqslant\theta\leqslant\pi} \Phi(x,\lambda)\,d\lambda = O\Big\{\int_{\eta}^{\pi} R^{\frac{1}{4}}e^{-(\delta/2\pi)R^{\frac{1}{4}}\theta}\,d\theta\Big\} = O(e^{-(\delta/2\pi)R^{\frac{1}{4}}\eta}). \tag{4.6}$$

The same result, of course, holds for the integral over $-\pi \leqslant \theta \leqslant -\eta$.

5. Next, we have

$$\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x,\lambda)\,d\lambda = \sum_{n=0}^{\infty} c_n\psi_n(x) \int_{Re^{-i\eta}}^{Re^{i\eta}} \frac{d\lambda}{\lambda-\lambda_n}.$$

This integral is bounded for all values of R and n (e.g. since we can take it along an arc of a circle with centre at $\lambda = \lambda_n$). Since $|\lambda-\lambda_n| \geqslant R-\lambda_n$, and the length of the contour is $2R\eta$, it is also

$$O\Big(\frac{R\eta}{|R-\lambda_n|}\Big).$$

Hence the above sum is

$$\sum_{\lambda_n \leqslant \frac{1}{2}R} O(|c_n| h_n^{-\frac{1}{2}} \eta) + \sum_{\frac{1}{2}R < \lambda_n \leqslant R - R\eta} O\left(\frac{|c_n| h_n^{-\frac{1}{2}} R\eta}{R - \lambda_n}\right) +$$

$$+ \sum_{R - R\eta < \lambda_n \leqslant R + R\eta} O(|c_n| h_n^{-\frac{1}{2}}) + \sum_{R + R\eta < \lambda_n} O\left(\frac{|c_n| h_n^{-\frac{1}{2}} R\eta}{\lambda_n - R}\right)$$

$$= S_1 + S_2 + S_3 + S_4.$$

Let $$\epsilon(r) = \Big\{\sum_{\lambda_n > r} c_n^2\Big\}^{\frac{1}{2}}.$$

Then $$S_3 = O\Big\{\epsilon(R - R\eta)\Big(\sum_{R - R\eta < \lambda_n \leqslant R + R\eta} h_n^{-1}\Big)^{\frac{1}{2}}\Big\}.$$

As in the case of $\sum_3$ above, we have

$$\sum_{R - R\eta < \lambda_n \leqslant R + R\eta} h_n^{-1} = O(R^{\frac{1}{2}}\eta).$$

Hence $$S_3 = O\{\epsilon(R - R\eta) R^{\frac{1}{2}} \eta^{\frac{1}{2}}\}.$$

Arguing as with $\sum_4$ and $\sum_2$, we obtain similarly

$$S_4 = O[\epsilon(R + R\eta) R^{\frac{1}{2}} \eta^{\frac{1}{2}}], \qquad S_2 = O[\epsilon(\tfrac{1}{2}R) R^{\frac{1}{2}} \eta^{\frac{1}{2}}].$$

Also $$S_1 = O\Big\{\eta\Big(\sum_{\lambda_n \leqslant \frac{1}{2}R} h_n^{-1}\Big)^{\frac{1}{2}}\Big\} = O\{\eta (R^{\frac{1}{2}} \log R)^{\frac{1}{2}}\},$$

as in the case of $\sum_1$. Altogether we obtain

$$\int_{Re^{-i\eta}}^{Re^{i\eta}} \Phi(x, \lambda)\, d\lambda = O(\eta R^{\frac{1}{2}} \log^{\frac{1}{2}} R) + O\{\epsilon(\tfrac{1}{2}R) R^{\frac{1}{2}} \eta^{\frac{1}{2}}\}.$$

Combining this with (4.6),

$$\int_{|\lambda| = R} \Phi(x, \lambda)\, d\lambda = O(e^{-(\delta/2\pi) R^{\frac{1}{2}} \eta}) + O(\eta R^{\frac{1}{2}} \log^{\frac{1}{2}} R) + O\{\epsilon(\tfrac{1}{2}R) R^{\frac{1}{2}} \eta^{\frac{1}{2}}\}.$$

Choosing η so that

$$R^{\frac{1}{2}} \eta \to \infty, \qquad \eta R^{\frac{1}{2}} \log^{\frac{1}{2}} R \to 0, \qquad \epsilon(\tfrac{1}{2}R) R^{\frac{1}{2}} \eta^{\frac{1}{2}} \to 0,$$

Theorem 1 follows.

6. Theorem 2 depends on the following lemma.

LEMMA 1. *Let $f(x)$ be of bounded variation over a finite interval (a, b), and zero elsewhere. Then*

$$c_n \psi_n(x) = O\left(\frac{1}{n}\right).$$

Let $f(x) = f_1(x) - f_2(x)$, where f_1 and f_2 are bounded and non-increasing in (a, b). Then

$$c_n = \int_a^b \psi_n(y) f_1(y)\, dy - \int_a^b \psi_n(y) f_2(y)\, dy.$$

By the second mean-value theorem

$$\int_a^b \psi_n(y)f_1(y)\,dy = f_1(a)\int_a^\beta \psi_n(y)\,dy \quad (a<\beta<b)$$

$$= -f_1(a)\int_a^\beta \frac{\psi_n''(y)}{\lambda_n - q(y)}\,dy$$

$$= -\frac{f_1(a)}{\lambda_n - q(\beta)}\int_\alpha^\beta \psi_n''(y)\,dy \quad (a<\alpha<\beta)$$

$$= \frac{f_1(a)}{\lambda_n - q(\beta)}\{\psi_n'(\alpha)-\psi_n'(\beta)\}$$

$$= O(\lambda_n^{-\frac{1}{4}}h_n^{-\frac{1}{2}})$$

by Lemma γ; and similarly for the other integral. Hence, by Lemma δ,

$$c_n\psi_n(x) = O(\lambda_n^{-\frac{1}{4}}h_n^{-1}).$$

Now* $$n \sim \frac{1}{\pi}\int_0^{h_n} \{\lambda_n - q(x)\}^{\frac{1}{2}}\,dx,$$

and in particular $n = O(\lambda_n^{\frac{1}{2}}h_n)$. This proves the lemma.

7. We now apply Wiener's general Tauberian theorem.† Let $a_n = c_n\psi_n(x)$, so that $a_n = O(1/n)$. Let

$$S = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

The theorem of the previous paper in this series shows that

$$-\lambda\Phi(x,-\lambda) = \sum_{n=0}^{\infty}\frac{\lambda a_n}{\lambda+\lambda_n} \to S$$

as $\lambda\to\infty$. Let $$S(x) = \sum_{\lambda_n\leqslant x} a_n.$$

Then

$$S(\lambda)+\lambda\Phi(x,-\lambda) = \sum_{\lambda_n\leqslant\lambda} a_n\left(1-\frac{\lambda}{\lambda+\lambda_n}\right) - \sum_{\lambda_n>\lambda}\frac{a_n\lambda}{\lambda+\lambda_n}$$

$$= O\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n(\lambda+\lambda_n)} + O\sum_{\lambda_n>\lambda}\frac{\lambda}{n(\lambda+\lambda_n)}$$

$$= O\left(\frac{1}{\lambda}\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n}\right) + O\left(\lambda\sum_{\lambda_n>\lambda}\frac{1}{n\lambda_n}\right).$$

* See my paper in *J. of London Math. Soc.* 19 (1944), 66–8.

† N. Wiener, *The Fourier Integral* (Cambridge, 1933), Ch. II.

Now

$$\sum_{\lambda<\lambda_n\leqslant 2\lambda}\frac{1}{n\lambda_n}\leqslant\frac{1}{\lambda}\sum_{\lambda<\lambda_n\leqslant 2\lambda}\frac{1}{n}=\frac{1}{\lambda}\sum_{N(\lambda)<n\leqslant N(2\lambda)}\frac{1}{n}$$

$$=O\left\{\frac{1}{\lambda}\frac{N(2\lambda)-N(\lambda)}{N(\lambda)}\right\}=O\left(\frac{1}{\lambda}\right)$$

by Lemma θ. Replacing λ by 2λ, 4λ,... and adding, we have

$$\sum_{\lambda_n>\lambda}\frac{1}{n\lambda_n}=O\left(\frac{1}{\lambda}\right).$$

Similarly,

$$\sum_{\lambda_n\leqslant\lambda}\frac{\lambda_n}{n}=O(\lambda).$$

Hence $\quad S(\lambda)+\lambda\Phi(x,-\lambda)=O(1), \qquad S(\lambda)=O(1).$

Also $\quad -\lambda\Phi(x,-\lambda)=\int_0^\infty\frac{\lambda}{\lambda+u}\,dS(u)=\int_0^\infty\frac{\lambda}{(\lambda+u)^2}S(u)\,du.$

Hence

$$S=\lim_{\lambda\to\infty}\int_0^\infty\frac{\lambda}{(\lambda+u)^2}S(u)\,du$$

$$=\lim_{\xi\to\infty}\int_{-\infty}^\infty\frac{e^\xi}{(e^\xi+e^\eta)^2}S(e^\eta)e^\eta\,d\eta$$

$$=\lim_{\xi\to\infty}\int_{-\infty}^\infty\frac{e^{\xi-\eta}}{(e^{\xi-\eta}+1)^2}S(e^\eta)\,d\eta$$

$$=\lim_{\xi\to\infty}\int_{-\infty}^\infty K_1(\xi-\eta)S(e^\eta)\,d\eta,$$

where $\quad K_1(x)=\dfrac{e^x}{(e^x+1)^2}.$

Now

$$\int_{-\infty}^\infty K_1(x)e^{-iux}\,dx=\int_{-\infty}^\infty\frac{e^{x-iux}}{(e^x+1)^2}\,dx=\int_0^\infty\frac{t^{-iu}}{(t+1)^2}\,dt$$

$$=\Gamma(1+iu)\Gamma(1-iu)\neq 0,$$

and

$$\int_{-\infty}^\infty K_1(x)\,dx=1.$$

Hence, by Wiener's Theorem 4,*

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_2(\xi-\eta)S(e^\eta)\,d\eta = S\int_{-\infty}^{\infty} K_2(x)\,dx,$$

where
$$K_2(\xi) = \begin{cases} 0 & (\xi<0),\\ e^{-\xi} & (\xi>0).\end{cases}$$

Thus

$$S = \lim_{\xi\to\infty}\int_{-\infty}^{\xi} e^{\eta-\xi}S(e^\eta)\,d\eta$$

$$= \lim_{x\to\infty}\frac{1}{x}\int_0^x S(y)\,dy.$$

Let δ be a number between 0 and 1, independent of x. Then

$$S = \frac{(1+\delta)S-S}{\delta} = \lim_{x\to\infty}\frac{1}{\delta x}\left\{\int_0^{(1+\delta)x} S(y)\,dy - \int_0^x S(y)\,dy\right\}$$

$$= \lim_{x\to\infty}\frac{1}{\delta x}\int_x^{(1+\delta)x} S(y)\,dy$$

$$= \lim_{x\to\infty}\left[S(x)+\frac{1}{\delta x}\int_x^{(1+\delta)x}\{S(y)-S(x)\}\,dy\right].$$

Now

$$S(y)-S(x) = \sum_{x<\lambda_n\leqslant y} a_n = \sum_{x<\lambda_n\leqslant y} O\left(\frac{1}{n}\right)$$

$$= O\left\{\sum_{x<\lambda_n\leqslant(1+\delta)x}\frac{1}{n}\right\} = O\left\{\sum_{N(x)<n\leqslant N\{(1+\delta)x\}}\frac{1}{n}\right\}$$

$$= O\left\{\frac{N\{(1+\delta)x\}-N(x)}{N(x)}\right\} = O(\delta)$$

by Lemma θ, provided that $x^{-\frac{1}{2}}\leqslant\delta$. Hence

$$\frac{1}{\delta x}\int_x^{(1+\delta)x}\{S(y)-S(x)\}\,dy = \frac{1}{\delta x}\int_x^{(1+\delta)x} O(\delta)\,dy = O(\delta).$$

Since δ is arbitrary, it follows that

$$\lim_{x\to\infty} S(x) = S.$$

* Loc. cit. 73–4.

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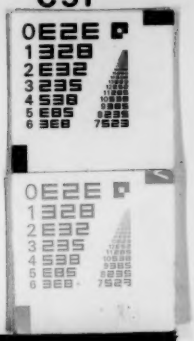
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